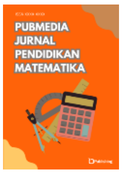

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Left and Right Derivatives of Summation Functions Limit and Eulerian Constants Induced by They

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Abstract: This study aims to investigate the existence and properties of one-sided derivatives of limit summation functions, particularly in relation to Euler-type constants, within the context of convex and concave real functions. It also seeks to generalize existing theorems related to the differentiability and summability of such functions. The research adopts a theoretical and deductive approach grounded in mathematical analysis. It begins with a comprehensive literature review of foundational concepts such as gamma and zeta functions, convexity, and Euler-Mascheroni constants. Utilizing formal mathematical reasoning, the study develops and proves several new theorems concerning the right and left derivatives of summation functions. The derived results are then validated through a series of examples involving known real functions, including convex and concave functions. The analysis confirms that under specific conditions, one-sided derivatives of summation functions exist and obey certain functional equations. Furthermore, the study demonstrates that sequences related to these derivatives converge under monotonicity assumptions. Applications include generalized inequalities and functional identities related to Euler's constant, gamma, and zeta functions. Ultimately, this research contributes to the understanding of marginal addition functions and offers new insights into the summability and differentiability of real functions involving Euler-type constants.

Keywords: One-sided Derivatives, Limit Summation Functions, Euler-type Constants

Introduction

The Euler-Mascheroni constants, γ and s, were introduced in the 18th century and are now considered some of the most widely recognized and useful mathematical constants. However, in 1997, a broader class of -Euler constants is investigated. Webster examined functions of the Γ form, which satisfy the Boher-Mollerup theorem, which generalizes the functional equation $j = (r+1)\ j(r)f(r)\ (r > 0)$. However, a novel idea known as the marginal addition function—an addition function for every function—was put forth by M. Hooushmand in 2001. It is defined on the portion of R that has every natural number in it. It demonstrates how clerical work might be regarded as its subsubject. whether in periodicals. He developed other related theories, including the Bohr-Möllrup main theorem, and we aim to elucidate certain singularity requirements for marginal addition functions. and How Functional Equations Relate to It This has been researched. We remember that

Schleicher and Müller employed a comparable tactic—using a series of functions of rational groups—to lessen assembly obstacles in 2010. More recently, functional and analytical groupings.

Methodology

This study is a theoretical investigation in the field of mathematical analysis that employs a deductive approach to formulate and prove theorems concerning one-sided derivatives of limit summation functions and their relationship to the Euler constant. The methods used include a literature review to gather fundamental concepts related to convex and concave functions, gamma and zeta functions, and the Euler constant from various scholarly references. Subsequently, a deductive-mathematical approach is applied by utilizing existing axioms, definitions, and theorems to develop new theorems and present formal proofs systematically. The study also involves symbolic and functional analysis of the properties of the examined functions, as well as comparisons with established theories. Finally, the proven theorems are validated through their application to several real function examples to assess their applicability to both convex and concave functions.

Result and Discussion

One-way derivatives of limit summation functions.

In this section, the conditions for the existence of one-way derivatives for some of the limit summation functions we will assess it. Specifically, the existence of one-way derivatives for summation functions we will check the limit when the function under study is convex or concave.

Definition 1. Suppose I is an interval and $j\,:\, I \to \mathbb{R}$ be a real function. If $ƙ \in I$ and limit.

$$\lim_{h\to 0^+} \frac{j(\eta + h) - j(\eta)}{h} \tag{1.1}$$

If it exists, we say that g is right derivable in s or j has a right derivative in s and that with the symbol $j'_+(r)$ we show if for every $j'_+(r), r \in I$ available function $j'_+ : I \to \mathbb{R}$ by the rule of $r \to j'_+(r)$ is called the (right) derivative function of j. Similar to this article, exists for the left derivative. Left derivative $j\ in\ s \in I$ with $j'_-(r)$ and function $j'_- : I \to \mathbb{R}$ by rule $r \to j'_-(r)$ is called the derivative (left) function of j. Whenever out of existence $j'_\pm$ we are talking about the assumption that both functions $j'_-\ and\ j'_+$ are available.

Definition 2. Let j be left or right differentiable on $\sum_j$ (for example, if j is convex or concave, these one-way derivatives exist). We explain.

$$j_{\sigma'_{c\pm}}(\eta) \coloneqq (j_{\sigma_n}(\eta))'_\pm = j(c) - \sum_{ƀ=1}^{c} j'_\pm(\eta + ƀ). \tag{1.2}$$

If the limit of this sequence is $m \in \sum_j$ be it available with the icon $j_{\sigma'_\pm}(ƙ)$ we show we also define:

$\gamma_{c\pm}(j, \eta) = -\, j_{\sigma'_\pm}(\eta)$

$$\gamma_{c\pm}(j) = -\, j_{\sigma'_\pm}(0) \tag{1.3}$$

If g is derived from T, that is $j'_+ = j'_-$ then.

$$j_{\sigma'_{c\pm}} = j_{\sigma'_c}(\eta) = -\gamma_{c\pm}(j,\eta) = -\gamma_c(j,\eta) \tag{1.4}$$

And if the sequels $j_{\sigma'_{c\pm}}$ be convergent, then

$$j_{\sigma'_{c\pm}}(\eta) = j_{\sigma'}(\eta) = -\gamma_{\pm}(j,\eta) = -\gamma(j,\eta). \tag{1.5}$$

Theorem 1. Suppose $j : (\delta, +\infty) \to \mathbb{R}$ *and* $\delta < 1$ function be convex, and $\eta(j,1) = 0$. In this case.

(A) – for each $j_{\sigma'_{c\pm}}(\eta), \eta \geq 0$ convergent and

$$\begin{aligned} -\gamma_+(j,1) \ \leq \ -\gamma_-(j,1) &\leq j_{\sigma'_+}(\eta) \leq j_{\sigma'_-}(\eta) \\ &\leq j_{\tilde{\sigma}}(\eta) \leq -\gamma_+(j) \leq -\gamma_-(j); \qquad (0 < \eta \leq 1). \end{aligned} \tag{1.6}$$

(B)- function $j_{\sigma'_\pm}(\eta)$ A solution to a functional equation

$$S(\eta) = j'_\pm(\eta) + \eta(\eta - 1) \tag{1.7}$$

Is.

Assume a proof q *and* $0 < \eta < 1$ be a natural number. Because j is convex, therefore given that $o < o + \eta < o + 1$ by applying the mean value theorem on the interval $[o, o + \eta]$, number $s_o \in (o, o + \eta)$ available so that

$$j'_-(\eta_b) \leq \frac{j(\eta + u) - j(b)}{s} \leq j'_+(\eta_u) \tag{1.8}$$

By multiplying the unequal sides, we get (1.8) in $-k$

$$-\eta j'_+(r_u) \leq j(b) - j(\eta + b) \leq -r\eta(\eta_b) \tag{1.9}$$

As a result

$$-\eta \sum_{b=1}^{c} j'_+(\eta_q) \leq \sum_{b=1}^{c} \big(j(b) - j(\eta + b)\big) \leq -\eta \sum_{b=1}^{c} j'_-(\eta_b) \tag{1.10}$$

So

$$\begin{aligned} \eta j(c) - \eta \sum_{b=1}^{c} j'_+(\eta_b) \ &\leq \eta j(c) + \sum_{b=1}^{c} (j(b) - j(\eta + b) \\ &\leq \eta j(c) - \eta \sum_{b=1}^{c} j'_-(\eta_b). \end{aligned} \tag{1.11}$$

By dividing the sides of the last relation by s. It is concluded that

$$j(c) - \sum_{b=1}^{c} j'_+(\eta_o) \leq \frac{j_{\sigma_c}(\eta)}{s} \leq j(c) - \sum_{b=1}^{c} j'_-(\eta_q) \tag{1.12}$$

Because j is convex, j'_+ *and* j'_- are ascending functions and $j'_-, \leq j'_+$ as a result

$$\begin{aligned} j'_-(b) \ \leq \ j'_+(b) &\leq j'_-(\eta_b) \leq j'_+(\eta_b) \\ &\leq \ j'_-(\eta + b) \leq j'_+(\eta + b) \leq j'_-(b + 1) \leq j'_+(b + 1) \end{aligned} \tag{1.13}$$

By summing the inequalities (1.13) per $1 \leq b \leq c$ and then add $j(c)$ on the sides of the resulting inequalities, we get.

$$\begin{aligned} j(c) - \sum_{b=1}^{c} j'_+(b + 1) &\leq j(c) - \sum_{b=1}^{c} j'_-(b + 1) \\ &\leq j(c) - j'_-(\eta + b) \\ &\leq j(c) - \sum_{b=1}^{e} j'_-(\eta + b) \end{aligned}$$

$$\leq j(c) - \sum_{\text{b}=1}^{c} j'_{+}(\eta_{\text{b}})$$
$$\leq j(c) - \sum_{\text{b}=1}^{c} j'_{-}(\eta_{\text{b}})$$
$$\leq j(c) - \sum_{\text{b}=1}^{c} j'_{+}(\text{b})$$
$$\leq j(c) - \sum_{\text{b}=1}^{e} j'_{-}(\text{b}). \tag{1.14}$$

Now according to the definition $j_{\sigma'_{c\pm}}(r)$ and we will have inequalities (1.12) and (1.14).

$$j_{\sigma'_{c+}}(1) \leq j_{\sigma'_{c+}}(\eta)$$
$$\leq j_{\sigma'_{c-}}(\eta)$$
$$\leq j(c) - \sum_{\text{b}=1}^{c} j'_{+}(\eta_{u})$$
$$\leq \frac{j_{\sigma_c}(\eta)}{\eta}$$
$$\leq j(c) - \sum_{\text{b}=1}^{c} j'(\eta_{\text{b}})$$
$$\leq j_{\sigma'_{c+}}(\text{b})$$
$$\leq j_{\sigma'_{c-}}(\text{b}) \tag{1.15}$$

As a result

$$\begin{aligned} -\gamma_{c+1}(j,1) &\leq -\gamma_{c-}(j,1) \\ &\leq j_{\sigma'_{c+}}(\eta) \\ &\leq j_{\sigma'_{c-}}(\eta) \\ &\leq j_{\tilde{\sigma}_c}(\eta) \\ &\leq -\gamma_{c+}(j) \\ &\leq -\gamma_{c-}(j) \end{aligned} \tag{1.16}$$

On the other hand, since j is $[1,+\infty)$ it is convex, therefore [2].

$$\begin{aligned} \frac{j(c+h)-j(c)}{h} &\leq j(c+1)-j(c) \\ &\leq \frac{j(c+1+h)-j(c+1)}{h} \end{aligned} \tag{1.17}$$

Where in $0 < h < 1$. Now with the assumption $h \to 0^{+}$, we get

$$j'_{+}(c) \leq -\text{K}_c(j,1) \leq j'_{+}(c+1) \tag{1.18}$$

Similarly with the assumption $-1 < h < 0$ convexity of function j on $[c+h, c, c+1+h, c+1]$ we get

$$j'_{-}(c) \leq -\text{Ŋ}_c(j,1) \leq j'_{-}(c+1) \tag{1.19}$$

Now we can conclude from the relations (1.18) and (1.19) that [3].

$$\begin{aligned} j'_{-}(c) &\leq j'_{+}(c) \\ &\leq -\text{Ŋ}_c(j,1) \\ &\leq -j'_{-}(c+1) \end{aligned}$$

$$\leq j'_{+}(c+!) \qquad (1.20)$$

As a result

$$\begin{aligned} \text{ɲ}_c(j'_{\pm},1) &= j'(c) - j'_{\pm}(c+1) \\ &\leq -j'_{\pm}(c+1) - \text{ɲ}_c(j,1) \\ &\leq \ 0 \end{aligned} \qquad (1.21)$$

We also have

$$\begin{aligned} j_{\sigma'_{c\pm}}(\eta) - j_{\sigma'_{(c+1)\pm}}(\eta) &= j(c) - \sum_{\text{b}=1}^{c} j'_{\pm}(\text{b}+\eta) - j(c+1) + \sum_{\text{b}=1}^{c} j'_{\pm}(\text{b}+\eta) \\ &= j(c) - j(c+1) + j'_{\pm}2(c+1+\eta) \end{aligned} \qquad (1.22)$$

As a result

$$\begin{aligned} j_{\sigma'_{(c+1)\pm}}(\eta) - j_{\sigma'_{c\pm}}(\eta) &= -\text{ɲ}_c(g,1) - j'_{\pm}(c+1+\eta) \\ &\leq -\text{ɲ}_c(j,1) - j'_{\pm}(c+1) \leq 0 \end{aligned} \qquad (1.23)$$

This means that for every $0 \leq \eta$, there is a functional sequence

$$j_{\sigma'_{c\pm}}(\eta) = -\gamma_{c\pm}(j,\eta)$$

It is descending. Now by putting $\eta = 0$ in the equation (1.23) and using the equation (1.21) to we get

$$\begin{aligned} \text{ɲ}_c(j'_{\pm},1) &= j'_{\pm}(c) - j'_{\pm}(c+1) \\ &\leq -\left(\text{ɲ}_c(j,1) + j'_{\pm}(c+1)\right) \\ &= j_{\sigma'_{(c+1)\pm}}(0) - j_{\sigma'_{c\pm}}(0) \\ &= -\gamma_{(c+1)s}(j) + \gamma_{c\pm}(j) \leq 0 \end{aligned} \qquad (1.24)$$

We also have

$$\begin{aligned} &j_{\sigma'_{c\pm}}(\eta) - j_{\sigma'_{c\pm}}(\eta-1) = j'_{\pm}(\eta) - j'_{\pm}(\eta+c), \\ &-\gamma_{c\pm}(j,1) = j'_{\pm}(1) - j'_{\pm}(c+1) - \gamma_{c\pm}(j) \end{aligned} \qquad (1.25)$$

Theorem 2. If $: [1,+\infty) \to \mathbb{R}$ have a monotonic derivative and $\text{K}(j,1) = 0$ then the sequel $j_{\sigma'_c}(\text{k})$ Roy $[0,+\infty)$ is convergent and if j' is ascending, then $j(\eta)$ *and* $j_{\tilde{\sigma}}(\eta)$ in inequality.

$$-\gamma(j,1) \leq j_{\sigma'}(\eta) \leq j_{\sigma'}(\eta) \leq -\gamma(j); \qquad (0 < \eta \leq 1) \qquad (1.26)$$

They apply also, if j' is downward, the direction of the above inequality is reversed. In addition, $j_{\sigma'}(\eta)$ a solution to a functional equation

$$\text{ɲ}(\eta) = j'(\eta) + \text{ɲ}(\eta-1) \qquad (\eta > 1) \qquad (1.27)$$

The result 1. Assume $j : (\delta,\infty) \to \mathbb{R}$ *and* $\delta < 1$ A convex (concave) function and sequence $\text{ɲ}_e(j,1)$ be bordered in this case.

(A)– sequel to $j_{\sigma'_{c\pm}}(\eta)$ is convergent for every $0 \leq \eta$ and if j is convex, then $j_{\sigma'_{\pm}}(\eta)$ in inequalities

$$j'_{\pm}(1) + \text{ɲ}(1) - \gamma_{\pm}(j) \leq j_{\sigma'_{\pm}}(\eta) \leq j_{\tilde{\sigma}}(\eta) \leq -\gamma_{\pm}(j), \qquad (0 < \eta \leq 1) \qquad (1.28)$$

And if j is concave, then

$$-\gamma_{\pm}(j) \leq j_{\tilde{\sigma}}(\eta) \leq j_{\sigma'_{\pm}}(\eta) \leq j'_{\pm}(1) + \eta(1) - \gamma_{\pm}(j), \qquad (0 < \eta \leq 1) \qquad (1.29)$$

(B)- function $j_{\sigma'_{\pm}}(\eta)$ a solution to a functional equation

$$\text{ɲ}(\eta) = j'_{\pm}(\eta) + \eta(j,1) + \text{ɲ}(r-1) \qquad (1.30)$$

Is Proof: because j is a convex function therefore $\text{ɲ}_c(j,1)$ it is uniform and because it is bounded assumption is as a result $\text{ɲ}_e(j,1)$ is convergent we put.

$$f(\eta) = j(\eta) + \text{ɲ}(1)\eta \qquad (\delta < \eta) \tag{1.31}$$

In this case

$$f'_{\pm}(\eta) = j'_{\pm}(\eta) + S(1)$$

And

$$f_{\sigma_c}(\eta) = j_{\sigma_c}(\eta).$$

Next, by presenting an example, we show that theorem 1. Is the generalization of theorem 2.

Example 1. function $g : (\frac{1}{2}, \infty) \to \mathbb{R}$ by rule.

$$j(\eta) = \begin{cases} \log & \eta \geq 2 \\ \frac{2}{3}s + log2 - \frac{4}{3}\frac{1}{2} < \eta \leq 2 \end{cases}$$

Consider.

(A) – show that this function is concave.

(B) – show using theorem 1.

$$\begin{aligned} \gamma - \frac{5}{3} &\leq \frac{log\Gamma(\eta+1)}{\eta} + \frac{\log(\eta+1)}{\eta} \\ &\leq \Psi(\eta+1) - \frac{1}{\eta+1} - \frac{2}{3} \\ &\leq \Psi(1) - \frac{1}{6} \\ &\leq -\gamma - \frac{1}{6} \end{aligned} \tag{1.32}$$

(C) – using theorem 1. Again, show:

$$\begin{aligned} -\gamma - \frac{17}{6} &\leq \frac{log\Gamma(\eta+1)}{\eta} + \frac{\log(\eta+1)}{\eta} \\ &\leq \Psi(\eta+1) - \frac{1}{\eta+1} - \frac{2}{3} \\ &\leq -\gamma - \frac{1}{6} \qquad (0 < \eta \leq 1). \end{aligned} \tag{1.33}$$

Solve (A) – from your previous knowledge in elementary math (by referring to the graph of the j function) easily it turns out that j is concave on its domain

(B) – we have

$$j_{\sigma'_c} + (\eta) = j(c) - \sum_{c=1}^{\infty} j'(\eta + c)$$
$$= \left(logc - \sum_{\text{ƀ}=1}^{c} \frac{1}{\eta + \text{ƀ}} \right) - \frac{1}{\eta+1} - \frac{2}{3}\left(\eta > \frac{1}{2}\right).$$

as a result

$$\begin{aligned} -\gamma_c(j) &= j_{\sigma'_c} + (0) \\ &= j(c) - \sum_{\text{ƀ}=1}^{c} j'_+(\text{ƀ}) \end{aligned}$$

$$\Psi = \left(logc - \sum_{ƅ=1}^{c} \frac{1}{ƅ} \right) - \frac{5}{3}$$

So

$$-\gamma_+(j) = \lim_{c\to\infty} \left(-\gamma_{c+}(j)\right) = -\gamma - \frac{5}{3} \tag{1.34}$$

Also

$$\begin{aligned}
j_{\sigma'_+}(ƙ) &= \lim_{c\to\infty} j_{\sigma'_c} + (\eta) \\
&= \lim_{c\to\infty} \left(1jc - \sum_{ƅ=1}^{e} \frac{1}{\eta + ƅ} \right) - \frac{1}{\eta+1} - \frac{2}{3} \\
&= \lim_{c\to\infty} \left[\left(loge - \sum_{ƅ=1}^{e} \frac{1}{ƅ} \right) - \left(\sum_{ƅ=1}^{e} \frac{1}{ƅ} - \sum_{ƅ=1}^{e} \frac{1}{r+ƅ} \right) \right] - \frac{1}{\eta+1} - \frac{2}{3} \\
&= \lim_{c\to\infty} \left[\left(logo - \sum_{ƅ=1}^{e} \frac{1}{ƅ} \right) + \left(\sum_{ƅ=0}^{e} \frac{1}{ƅ+1} - \sum_{ƅ=0}^{e} \frac{1}{ƅ+\eta+1} \right) \right] - \frac{1}{\eta+1} - \frac{2}{3} \\
&= -\gamma + \sum_{ƅ=0}^{\infty} \left(\frac{1}{ƅ+1} - \frac{1}{ƅ+\eta+1} \right) - \frac{1}{ƙ+1} - \frac{2}{3} \\
&= \Psi(ƙ+1) - \frac{1}{\eta+1} - \frac{2}{3} \qquad \left(\eta > \frac{1}{2}\right)
\end{aligned}$$

So

$$j_{\sigma'_+}(\eta) = \Psi(s+1) - \frac{1}{\eta+1} - \frac{2}{3} \qquad (\eta > \frac{1}{2}) \tag{1.35}$$

In addition

$$\begin{aligned}
j_{\sigma_c}(\eta) &= \eta g(c) + \sum_{ƅ=1}^{c} \left(j(ƅ) - j(\eta + ƅ) \right) \\
&= \eta\, logc + \sum_{ƅ=1}^{c} (\log ƅ - \log{(\eta + ƅ)}) \\
&= \eta \log c + \sum_{ƅ=1}^{c} log \frac{ƅ}{\eta + ƅ} \\
&= \log c\eta + \log \prod_{ƅ=1}^{c} \frac{ƅ}{\eta + ƅ} \\
&= log\ c^{\eta} + log \frac{1 \cdot 2 \cdot \ldots \cdot c}{(\eta+1) \ldots (\eta + c)} \\
&= \log\ c^{\eta} + log \frac{c!}{(\eta+1)(\eta+2) \ldots (\eta+c)} \\
&= log \frac{c!\, c^{\eta}}{(\eta+1) \ldots (\eta+c)}
\end{aligned}$$

As a result

$$\begin{aligned}
j_{\sigma}(\eta) &= \lim_{c\to\infty} j_{\sigma_c}(\eta) \\
&= \lim_{c\to\infty} \frac{c!\, c^{\eta}}{(\eta+1) \ldots (\eta+c)}
\end{aligned}$$

$$= \log \lim_{c\to\infty} \frac{c!\, c^{\eta}}{(\eta+1)\ldots(\eta+c)}$$
$$= \log \lim_{c\to\infty} \frac{c!\, c^{\eta} c}{(\eta+1)\ldots(\eta+c)(\eta+c+1)} \frac{\eta+c+1}{c}$$
$$= log\left(\lim_{c\to\infty} \frac{c!\, c^{\eta+1}}{(\eta+1)\ldots(\eta+c+1)} \cdot \lim_{n\to\infty} \frac{c+\eta+1}{c}\right)$$
$$= \log\big(\Gamma(\eta+1)\cdot(\eta+1)\big)$$
$$= \log \Gamma(\eta+1) + \log(\eta+1)$$

The result is that

$$j_{\sigma}(\eta) = log\Gamma(\eta+1) + 1j(\eta+1) \qquad (1.36)$$

As a result

$$j_{\tilde{\sigma}}(\eta) = \frac{j_{\sigma}(\eta)}{\eta} = \frac{log\Gamma(\eta+1)}{\eta} + \frac{\log(\eta+1)}{\eta} \qquad (1.37)$$

Now, theorem 1. Requires that [4].

$$-\gamma_{+}(j) \le j_{\tilde{\sigma}}(\eta) \le j_{\sigma'_{+}}(\eta) \le -\gamma_{+}(j.1) \qquad (1.38)$$

Relationship now.

$$\Psi(\eta+1) = \frac{1}{s} + \Psi(\eta)$$

And relations (1.34),(1.35),(1.36),(1.37) and (1.38) result that

$$-\gamma - \frac{5}{3} \le \frac{log\Gamma(\eta+1)}{\eta} + \frac{\log(\eta+1)}{\eta}$$
$$\le \Psi(\eta+1) - \frac{1}{\eta+1} - \frac{2}{3}$$
$$\le \Psi(2) - \frac{1}{2} - \frac{2}{3}$$
$$= \Psi(1) - \frac{1}{6}. \qquad (1.39)$$

(C) Quite similar to part (B) it can be seen that.

$$-\gamma_{-}(g) = -\gamma - \frac{17}{6}$$
$$j_{\sigma'_{-}}(\eta) = \Psi(\eta+1) - \frac{1}{\eta+1} - \frac{2}{3}$$
$$j_{\tilde{\sigma}}(\eta) = \frac{log\Gamma(\eta+1)}{\eta} + \frac{\log(\eta+1)}{\eta}$$

And as a result

$$-\gamma - \frac{17}{6} \le \frac{log\Gamma(\eta+1)}{\eta} + \frac{\log(\eta+1)}{\eta}$$
$$\le \Psi(\eta+1) - \frac{1}{\eta+1} - \frac{2}{3}$$
$$\le -\gamma - \frac{1}{6} \qquad (0 < \eta \le 1)$$

$$j(\eta) = \begin{cases} -\sqrt{\eta - 1} & \eta \geq 2 \\ \\ -\eta + 1 & \frac{1}{2} < \eta \leq 2 \end{cases}$$

Consider.

(A) show that this function is convex.

(B) show using theorem 1.

$$\begin{aligned} & 1 - \frac{1}{2}\sum_{\mathfrak{b}=1}^{\infty} \frac{2\sqrt{\mathfrak{b}-1} - \sqrt{\mathfrak{b}} - \sqrt{\mathfrak{b}-1}}{\sqrt{\mathfrak{b}+1}(\sqrt{\mathfrak{b}} + \sqrt{\mathfrak{b}-1})} \\ & \leq 1 - \frac{1}{2}\sum_{\mathfrak{b}=1}^{\infty} \frac{2\sqrt{\mathfrak{b}-1} - \sqrt{\mathfrak{b}} - \sqrt{\mathfrak{b}-1}}{\sqrt{\mathfrak{b}+1}(\sqrt{\mathfrak{b}} + \sqrt{\mathfrak{b}-1})} \\ & \leq \lim_{c\to\infty}\left(-\sqrt{c-1} + \sum_{\mathfrak{b}=1}^{c} \frac{1}{\sqrt{\eta + \mathfrak{b} - 1} + \sqrt{\mathfrak{b}-1}}\right) \\ & \leq 1 + \frac{1}{2}\zeta\left(\frac{1}{2}\right) \qquad\qquad (0 < \eta \leq 1) \end{aligned} \tag{1.40}$$

Solve it is quite similar to the previous example. (A) by drawing the graph of j, it is easy to seethat j is convex.

(B) – we have

$$\begin{aligned} j_{\sigma_c}(\eta) &= \eta j(c) + \sum_{\mathfrak{b}=1}^{c}\big(j(\mathfrak{b}) - j(\eta + \mathfrak{b})\big) \\ &= -\eta\sqrt{c-1} + \sum_{\mathfrak{b}=1}^{c}\left(\sqrt{\eta + \mathfrak{b} - 1} - \sqrt{\mathfrak{b}}\right) \\ &= -\eta\sqrt{c-1} + \sum_{\mathfrak{b}=1}^{c} \frac{\eta}{\sqrt{\eta + \mathfrak{b} - 1} + \sqrt{\mathfrak{b}}} \end{aligned}$$

In result

$$j_{\tilde{\sigma}}(\eta) = \frac{j_\sigma(\eta)}{\eta}$$

$$= \lim_{c\to\infty}\left(-\sqrt{c-1} + \sum_{\mathfrak{b}=1}^{c} \frac{1}{\sqrt{\eta + \mathfrak{b} - 1} + \sqrt{\mathfrak{b}}}\right)$$

And

$$\begin{aligned} j_{\sigma'_{c}\pm}(\eta) &= j(c) - \sum_{\mathfrak{b}=1}^{c} j'_{\pm}(\eta + \mathfrak{b}) \\ &= -\sqrt{c-1} + \frac{1}{2}\sum_{\mathfrak{b}=1}^{c} \frac{1}{\sqrt{\eta + \mathfrak{b} - 1}} \end{aligned}$$

In result

$$\begin{aligned} j_{\sigma\pm}(\eta) &= \lim_{c\to\infty} j_{\sigma'_{c\pm}}(\eta) \\ &= 1 - \frac{1}{2}\sum_{\mathfrak{b}=1}^{\infty} \frac{2\sqrt{\mathfrak{b} + \eta} - (\sqrt{\mathfrak{b}} - \sqrt{\mathfrak{b}-1})}{\sqrt{\mathfrak{b} + \eta}(\sqrt{\mathfrak{b}} + \sqrt{\mathfrak{b}-1})} \end{aligned}$$

And so

$$j_{\sigma'}(1) = -\gamma(j,1)$$
$$= 1 - \frac{1}{2}\sum_{ƀ=1}^{\infty} \frac{2\sqrt{ƀ+\eta} - (\sqrt{ƀ} - \sqrt{ƀ-1})}{\sqrt{ƀ+\eta}(\sqrt{ƀ} + \sqrt{ƀ-1})}$$

And

$$j_{\sigma'}(ƀ) = -\gamma(j)$$
$$= 1 - \frac{1}{2}\sum_{ƀ=1}^{\infty} \frac{2\sqrt{ƀ+\eta} - \left(\sqrt{ƀ} - \sqrt{ƀ-1}\right)}{\sqrt{ƀ+\eta}\left(\sqrt{ƀ} + \sqrt{ƀ-1}\right)}$$
$$= 1 - \frac{1}{2}\sum_{ƀ=1}^{\infty} (ƀ = 1)\sqrt{ƀ}$$

Now according to theorem 1.

$$-\gamma(j,1) \le j_{\sigma'}(\eta) \le j_{\tilde{\sigma}}(\eta) \le -\gamma(\mathrm{j})$$

In result

$$1 - \frac{1}{2}\sum_{ƀ=1}^{\infty} \frac{2\sqrt{ƀ+\eta} - (\sqrt{ƀ} - \sqrt{ƀ-1})}{\sqrt{ƀ+\eta}(\sqrt{ƀ} + \sqrt{ƀ-1})}$$
$$\le 1 - \frac{1}{2}\sum_{ƀ=1}^{\infty} \frac{2\sqrt{ƀ+\eta} - (\sqrt{ƀ} - \sqrt{ƀ-1})}{\eta(\sqrt{ƀ} + \sqrt{ƀ-1})}$$
$$\le \lim_{c\to\infty}\left(-\sqrt{c-1} + \sum_{ƀ=1}^{c} \frac{1}{\sqrt{\eta+ƀ-1} + \sqrt{ƀ-1}}\right)$$
$$\le 1 + \frac{1}{2}\zeta\left(\frac{1}{2}\right) \qquad (0 < \eta \le 1).$$

Suppose $(d_c)_1^\infty$ be a series of actual numbers in which.

(A) - (d_c) is ascending.

(B) – for each $d_c < \frac{d_{c-1}+d_{c+1}}{2}, c$ (means d_c is strictly convex).

(C) - $\lim_{c\to\infty}(d_{c+1} - d_c) = 0$ sequence of functions $j_c : [c, c+1) \to \mathbb{R}$ by rule $g_c(\eta) = (d_{c+1} - d_c)(\eta - c) = d_c$ we define and then the function $j : [1, +\infty) \to \mathbb{R}$ by rule.

$$j(\eta) = \sum_{c=1}^{\infty} j_c(\eta)\eta_{[c,c+1]}(\eta) \qquad (1.41)$$

We explain. Intuitively, the graph j is the set of line segments that go from the points $D_1(1, d_1)$ *and* $D_1(2, d_2)$…passes. It is clear that j is not differentiable at correct points and therefore the theorem 1. Is not applicable to it and we can apply theorem 1. About it. And from the last theorem, it can be easily seen that j is summable and inequalities (1.6) for it is established.

More examples and inequalities

In this section, using theorem 2. We prove two important inequalities that these inequalities they have already been proven in reference by leforgia and natalifi with another and laborious method.

Example 2. Function $j : [1, \infty) \to \mathbb{R}$ by rule $j(\eta) = \log m$ considered and using show from theorem 2.

$$-1 + \gamma \leq \frac{-\log \Gamma\,(\eta + 1)}{\eta} \leq \gamma(log, \eta) \leq \gamma \qquad 0 < \eta \leq 1 \tag{1.42}$$

And draw conclusions from it

$$c^{-\gamma\eta} \leq c^{\eta\Psi(\eta+1)} \leq \Gamma(\eta + 1) \leq c^{(1-\gamma)\eta} \qquad (0 < \eta \leq 1)$$

Solve according to example 2. We see that

$$log_{\sigma}(\eta) = \log \Gamma(\eta + 1)$$
$$log_{\sigma'}(\eta) = -\gamma(log, \eta) = \Psi(\eta + 1)$$
$$\gamma = \gamma(log, 0)$$

In result

$$-\gamma(j, 0) = -\Psi(1) = -1 + \gamma$$
$$j_{\tilde{\sigma}}(\eta) = \frac{j_{\sigma}(\eta)}{\eta} = \frac{-\log \Gamma(\eta + 1)}{\eta}$$
$$j_{\sigma'}(\eta) = \gamma(log, \eta)$$
$$j'(1) + \jmath(1) - \gamma(j) = 1 + 0 - 1 + \gamma = \gamma$$

Now according to inequality

$$-\gamma(j) \leq j_{\tilde{\sigma}}(\eta) \leq j_{\sigma'}(\eta) \leq j'(1) + \jmath(1) - \gamma(j), \quad (0 < \eta \leq 1)$$

Will have

$$-1 + \gamma \leq \frac{-\log \Gamma(\eta + 1)}{\eta} \leq \gamma(log, \eta) \leq \Gamma, \quad (0 < \eta \leq 1).$$

Now, by multiplying the sides of the last inequality, we will have $-s$ [8].

$$-\gamma\eta \leq -\eta\,\gamma(\log, \eta) \leq \log \Gamma(\eta + 1) \leq \eta(1 - \gamma)$$

In result

$$c^{-\gamma\eta} \leq c^{\eta\Psi(\eta+1)} \leq \Gamma(\eta + 1) \leq c^{(1-\gamma)\eta}.$$

The first part of the next example, inequalities for the zeta function and $(v, \eta)\ \zeta$ for the case that 1< v and gives $0 < \eta \leq 1$

Example 3. Suppose m is a constant real number. Function by rule $j(\eta) = \eta^{s}$ we explain. According to three different modes, $s = -1, s < 0$ and $0 < s < 1$ using theorem 1. Show [8].

(A)

$$\begin{aligned} s\zeta(1 - s) &\leq \frac{1}{\eta}\sum_{c=1}^{\infty} \frac{1}{(c + \eta)} - \frac{1}{c - s} \\ &\leq s\left(\zeta(1 - s, \eta) - \frac{1}{\eta^{1-s}}\right) \\ &\leq s\zeta(1 - \mathrm{s}) - \mathrm{s} \end{aligned} \tag{1.43}$$

(B)

$$\begin{aligned} s + \frac{\pi^2}{6} &\leq \sum_{c=1}^{\infty} \frac{1}{(c + \eta)^2} \\ &\leq \sum_{c=1}^{\infty} \frac{1}{c^2 + c\eta} \\ &\leq \frac{\pi^2}{6} \qquad (0 < \eta \leq 1) \end{aligned} \tag{1.44}$$

(C)

$$1-\sum_{c=1}^{\infty}((c-1)^s-c^s-sc^{s-1})$$
$$\leq 1-s(\eta+1)^{s-1}-\sum_{c=1}^{\infty}((c-1)^s-c^s-s(\eta+)^{s-1})$$
$$\leq \frac{1}{r}\sum_{c=1}^{\infty}\left((1+\eta)c^s-(c+\eta)^s-v(c-1)^s\right)$$
$$\leq 1-s-\sum_{c=1}^{\infty}((c-1)^s-c^s-sc^{s-1}). \tag{1.45}$$

Solve (A). in this case $\jmath(1)=0$ *and* $j''(\eta)>0$ *and* $j'(\eta)<0$ because $j''(\eta)>0$ *so* j' is ascending and according to the theorem 1. We have [9].

$$-\gamma(j,1)\leq j_{\sigma'}(\eta)\leq j_{\tilde{\sigma}}(\eta)\leq -\gamma(j), \qquad (0<\eta\leq 1) \tag{1.46}$$

We have now

$$j_{\sigma_c'}(s)=\frac{1}{c^{-\eta}}-\eta\sum_{\mathrm{b}=0}^{c}\frac{1}{(s+\mathrm{b})^{1-\eta}}+\frac{\eta}{s^{1-\eta}}$$

So

$$j_{\sigma'}=s\left(\frac{1}{r^{1-s}}-\zeta(1-s,\eta)\right) \tag{1.47}$$
$$-\gamma(\mathrm{j},1)=\mathrm{s}\left(1-\zeta(1-\mathrm{s})\right) \tag{1.48}$$
$$-\gamma(j,1)=-\gamma(j)=j_{\sigma'}(0)=\lim_{c\to\infty}\left(c^s-\sum_{\mathrm{b}=1}^{c}su^{s-1}\right)$$
$$=s\sum_{\mathrm{b}=1}^{\infty}\frac{1}{u^{1-s}}=s\zeta(1-s)=2(2\pi)^{-s}\Gamma(s+1)\cos\left(\frac{\pi}{2}s\right)\zeta(s) \tag{1.49}$$

Now, from the relations (1.46) ,(1.47), (1.48) and (1.49) the relation (1.43) is obtained. (to this is the case that after placing the obtained values in (1.47) ,(1.48) ,(149) in (1.46) and then multiplying the sides by a negative one, the verdict is obtained).

(B) - $r=-1$. in this case $j(\eta)=\frac{1}{\eta}$ so

$$j'(\eta)=-\frac{1}{\eta^2}<0, \qquad j''(\eta)=\frac{1}{\eta^3}>0$$

Therefore, j' is ascending again and we have

$$j_{\sigma_c}(\eta)=\frac{\eta}{c}+\sum_{\mathrm{b}=1}^{c}\left(\frac{1}{\mathrm{b}}-\frac{1}{\mathrm{b}+\eta}\right)$$
$$=\frac{\eta}{c}+\eta\sum_{\mathrm{b}=1}^{c}\frac{1}{\mathrm{b}^2+u\eta}$$

So

$$j_{\sigma}(\eta)=\eta\sum_{\mathrm{b}=1}^{\infty}\frac{1}{\mathrm{b}^2+\mathrm{b}\eta}$$

In result

$$j_{\tilde{\sigma}}(ɳ) = \frac{j_{\sigma}(ɳ)}{ɳ} = \sum_{ƅ=1}^{\infty} \frac{1}{ƅ^2 + ƅɳ}$$

Also

$$j_{\sigma_c'}(ɳ) = j(c) - \sum_{ƅ=1}^{c} j'\,(v + ƅ)$$

$$= \frac{1}{c} + \sum_{ƅ=1}^{e} \frac{1}{(ɳ + ƅ)^2}$$

Therefore

$$j_{\sigma'}(ɳ) = \sum_{ƅ=1}^{\infty} \frac{1}{(ɳ + ƅ)^2}$$

In result

$$-\gamma(j) = -\gamma(j, 0) = j_{\sigma'}(0) = \sum_{ƅ=1}^{\infty} \frac{1}{ƅ^2} = \frac{\pi^2}{6}$$

$$-\gamma(j, 1) = j_{\sigma'}(1) = \sum_{ƅ=1}^{\infty} \frac{1}{(1 + ƅ)^2} \frac{\pi^2}{6} - 1$$

Therefore, it follows from the relationship (1.46) that [8].

$$\frac{\pi^2}{6} - 1 \leq \sum_{ƅ=1}^{\infty} \frac{1}{(ƅ + ɳ)^2} \leq \sum_{ƅ=1}^{\infty} \frac{1}{ƅ^2 + ƅ^2 ɳ} \leq \frac{\pi^2}{6} \qquad 0 < ɳ \leq 1$$

(C) – in this case $0 < s < 1$.

$$j(ɳ) = ɳ^s$$

$$j'(ɳ) = sɳ^{s-1} > 0$$

$$j''(ɳ) = s(s-1)ɳ^{s-2} = \frac{s(s-1)}{ɳ^{2-s}} < 0$$

Therefore, g is decreasing and

$$j_{\sigma_c'}(ɳ) = c^s - \sum_{ƅ=1}^{c} s(ɳ + ɳ)^{s-1}$$

$$= c^s - s(ɳ + 1)^{s-1} - \sum_{ƅ=2}^{c} s(ɳ + ƅ)^{s-1}$$

In result

$$j_{\sigma'}(ɳ) = 1 - s(ɳ + 1)^{s-1} - \sum_{ƅ=2}^{\infty} ((u-1)^s - ƅ^s - s(ɳ + ƅ)^{s-1})$$

So

$$-\gamma(j) = j_{\sigma'}(0) = 1 - s - \sum_{ƅ=2}^{\infty} ((ƅ - 1)^s - ƅ^s - sƅ^{s-1})$$

$$-\gamma(j, 1) = j_{\sigma'}(1) = 1 - \sum_{o=2}^{\infty} ((c-1)^s - c^s - sc^{s-1})$$

$$j_{\sigma_c}(\eta) = \eta c^s + \sum_{b=1}^{c} (b^s - (b+\eta)^s)$$

In result

$$j_{\tilde{\sigma}}(\eta) = \frac{1}{\eta} \sum_{c=1}^{\infty} \left((1+\eta)c^s - (c+\eta)^s - \eta(c-1)^s \right)$$

Now the verdict is obtained from theorem 1.

Conclusion

This research report includes the study of Kama and zeta, marginal addition functions and their relationships, and Euler-type constants. Through the research article's theories and examples, explain the relationship and practical application between the derivation and Euler's constant.

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