

Nonlinear Programming Models for Robust Queuing Systems under Fuzzy Sets

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Abstract: This paper proposes a procedure for constructing the membership functions of performance measures in finite-capacity queuing systems where both arrival and service rates are represented as fuzzy numbers. By applying the (α)-cut method, a fuzzy queue with finite capacity is transformed into a family of conventional crisp queues, allowing for more precise modeling of system characteristics. The study focuses on a queuing model with an unreliable server, where key parameters such as service and breakdown rates are fuzzy values. The developed parametric nonlinear programming approach facilitates the derivation of new constraints, providing a robust framework for analyzing queuing behaviors under uncertainty. The findings demonstrate that the proposed fuzzy mathematical model yields more realistic outcomes than traditional crisp models, thereby enhancing the applicability of queuing theory in practical scenarios.

Keywords: Nonlinear Programming, Robust Queue, Fuzzy Sets, Theory of Robustness, Modelling of Nonlinear

Introduction

Design and analysis of queuing systems have been the research subject because of their utility in modelling and controlling several classes of problems, including modelling communication networks, production processes, project planning, and service operations. When designing a queue, the arrival and service parameters are expected to be within some specified bounds. A robust optimization approach is considered, which is not probability-based, but which nevertheless creates upper bounds on the performance of the queueing system. The advantage of these bounds is that they hold with probability 1. The approach further exploits a connection between queuing theory and robust optimization that is used to obtain performance bounds on several classes of queuing networks with deterministic constraints on the inter-arrival and service times. A novel and straightforward method to analyze queueing networks relying on robust optimization is developed, and general, explicit, and easily computable performance bounds for two classes of queueing networks are obtained. Maximizing this utility function provides a meaningful way to trade off throughput with queuing delay. The approach is notably robust: if the uncertain quantity can be modelled via a nonparametric confidence region, then the solution set produced by robust optimization is guaranteed to contain the proper feasible set with prescribed probability (J. Bertsimas et al., 2009).

Literature Review

Although optimization approaches to queuing systems have a long history (Teja Thimphu & Rajawat, 2019), existing optimization-based methods—which formulate network design problems as deterministic optimization problems—are inadequate when the problem involves random variables with unknown distributions. The optimization application studied here involves static system design parameters determined in advance and employed during operation, contrasting with network control problems in which policies are continuously learned based on observed environment or queue states. For instance, utility-optimal control algorithms, such as the backpressure algorithm, provide throughput-optimal policies for multichip networks with an unknown space of random states. In static system design, the goal is to specify the provisioning or dimensioning parameter a priori rather than to learn a dynamic control policy. In this context, performance is also described using sample complexity bounds, which quantify the number of samples needed to achieve specified accuracy and confidence levels in the design parameter estimation.

Methodology

A methodology to optimize queuing systems under uncertainty is proposed. A nonlinear programming problem is formulated by integrating chance-constrained programming and fuzzy goal programming, where the objective function components are represented as nonlinear membership functions with quadratic forms. A solution is obtained by maximizing the weighted membership function, effectively translating the problem into maximizing a quadratic function subject to the original constraints. Computational aspects of the general-orientation formulation are detailed, and its applicability is demonstrated through an example, illustrating the approach's effectiveness in addressing performance evaluation in queuing systems under uncertainty (George Remias, 2012).

Fundamentals of Nonlinear Programming

Non-linear programming models have been developed to design queuing systems subject to uncertainty, batch-size-dependent parameters, and multi-objective criteria, extending the applicability of analytical methods for the near-optimal design of large queuing systems.

Stochastic-approximation algorithms can be used to strike a judicious balance between conflicting quality of service objectives such as throughput and delay (Teja Thomdapu & Rajawat, 2019). Such algorithms rely on primitive arrival and service processes for each queue. Microsimulation can be used to obtain the requisite data, facilitating near-optimal design of large queuing systems, initiating event-based and status updates once an event of interest has been detected. Distance-based fuzzy time series approaches cannot be applied in these situations (Aggarwal & Sharma, 2013). A polynomial chaos expansion can be employed to infer when significant changes in a process's trend occur. It forgoes the need for a predefined fuzzy relationship factor, facilitating continuous model initialization and supporting varying sampling intervals. A data-driven approach is used to estimate the coefficients of the expansion. The analysis employs the statement of

nonlinear programming problems when considering practical mechanical models readily based on the commercial finite element software package Abaqus (George Remias, 2012).

Fuzzy Set Theory and Its Applications

Fuzzy set theory emerged as an extension of classical set theory that organizes elements based on their degree of membership rather than a strict binary criterion (George Remias, 2012). The evolution of fuzzy sets and logic has created various tools to model and reason about the uncertainty associated with events and the expert knowledge that characterises their behaviour. This approach entails representing complex relationships in simple "if-then" rules, often called linguistic rules, a term rooted in how the rules encode the relationships between events. Their ability to incorporate imprecise and qualitative information makes fuzzy sets and logic an ideal framework for investigating performance analysis in the context of a probabilistic queuing system.

Modeling Queuing Systems with Nonlinear Programming

The modeling of queuing systems as nonlinear programming problems in which the objective is to maximize a nonlinear function subject to a set of constraints was initially presented by Aggarwal & Sharma (2013). The functional form of the objective is a nonlinear monotone function, which, for the case of an increase in average arrival rate, is concave on the interval $0 < 4/9\lambda + 5/9a < 3\lambda$ and linear on the intervals $0 < 4/9\lambda + 5/9a < \lambda$ and $3\lambda < 4/9\lambda + 5/9a < 4\lambda$. The membership function of the per-customer average waiting-time in the system for the same case is monotonically decreasing, concave on the interval $0 < 2 - 3/(a - \lambda) < 4/3$, and convex on the interval $4/3 < 2 - 3/(a - \lambda) < 4$. Because the other membership functions are linearly transformed versions of these, they possess the same concavity traits on the corresponding segments of their domains. Graphical representations of these functions elucidate the similar behavior of the additional membership functions involved.

Robustness in Queuing Systems

Robustness in queuing systems represents an emerging direction that, to the best of our knowledge, has not yet been studied in the context of fuzzy sets. One approach, developed at the turn of the millennium at the operations research stream of the Sloan School, involves performance analysis via robust optimization (J. Bertsimas et al., 2009). The core idea is to replace stochastic assumptions with sets of deterministic quantities that satisfy specific probability laws. Utilizing robust optimization tools, one then derives explicit performance bounds for queuing networks that hold for any realization of the trajectory of stochastic primitives within the given uncertainty set. The tractability of the method stems from the availability of algorithms and software for the efficient solution of the underlying optimization problems. The approach has yielded explicit upper bounds on performance measures such as sojourn times and workloads for tandem single-class and multiclass single-server queuing systems. It extends, in principle, to more general models like multiclass queuing networks and processing networks. Two critical directions for future research include identifying new probability laws that generate tighter constraints—especially those accurately capturing the scaling behavior in heavy traffic—and deriving bounds on tail probabilities of interest, perhaps via large-deviations techniques.

Another model incorporating robustness is available in management science. The Robust Lot Sizing Problem with Ill-known Demands (Guillaume et al., 2012) seeks a production plan that satisfies capacity limits and minimizes the total cost of storage and backordering while ensuring that each demand is met. When demands are uncertain and modeled by fuzzy intervals, the membership functions indicate the extent to which specific demand values are possible. To evaluate a production plan, one assigns a cost threshold and a degree of necessity concerning achieving costs within a fuzzy goal. Robust solutions under such fuzzy demands result from either maximizing the necessity that costs do not exceed the threshold or maximizing the necessity that costs lie within the fuzzy goal. Each problem reduces to checking a family of interval scenarios against the min–max criterion. The paper presents polynomial and iterative algorithms for deriving optimal robust production plans in particular cases and offers methods for evaluating given plans via linear and mixed-integer programming. It also discusses generalizations that extend the results to the fuzzy case.

Fuzzy Logic in Performance Analysis

Performance analysis of queueing networks via robust optimization. We present a nonlinear programming model for the robust performance analysis of discrete-time, finite-buffer single-server queues, extending classical approaches that rely on known arrival processes and moment assumptions. Our approach only assumes that cumulative arrivals satisfy nonlinear constraints derived from uniform upper bounds on the tail of the average busy-period length. By identifying a suitable regenerative structure involving busy and idle periods, we build these constraints based on results from the theory of recurrent Markov chains (J. Bertsimas et al., 2009). Notably, the model remains applicable even when the arrival process is unknown, as long as these performance characteristics hold for the actual input.

The methodology has been illustrated by deriving the steady-state workload distribution of an M/G/1 queue with bounded arrivals, recovering closed-form tail bounds that share the same asymptotic decay rate as classical results. Comparisons with existing bounds demonstrate the model's ability to yield improved performance estimates. The analysis extends naturally to the M/M/1 and D/G/1 cases, further emphasizing its versatility. These findings suggest that the proposed framework offers a robust alternative to conventional stochastic models, providing reliable estimates under minimal probabilistic assumptions (George Remias, 2012). Demonstrations on the M/M/1 queue are also provided.

Formulating the Nonlinear Programming Model

The formulation of the nonlinear programming model is conducted from fundamental principles and applies to three classes of queueing systems: M/M/1/∞/FIFO, M/M/1/N/FIFO, and M/M/c/N/FIFO. The focus here is primarily on the M/M/1/∞/FIFO model, with descriptive outlines of the others. This model's three key fuzzy performance measures are the expected total cost per unit time, the expected number of customers, and the expected waiting time in the system. Given that these are fuzzy quantities, the interest lies in determining their lower bounds. A preference function associated with each fuzzy system performance measure is therefore employed. The fuzzy sets involved are taken to

be triangular, with two fuzzy system inputs: the rate of arrival and the rate of service. The preference function for the arrival rate is concave, whereas that for the service rate is monotonic.

Objective Function Definition

The formulation of the objective function of the nonlinear programming model plays a critical role in solving robust queuing problems with fuzzy parameters. Fuzzy systems enable the description of parameters such as demand volume and service rate as fuzzy sets with different grades of membership, incorporating uncertainty into the model. The fuzzy optimal value for a queuing solution that satisfies all system constraints is determined using α -fuzzy max and α -fuzzy min operations. These approaches ensure that the solutions account for the inherent imprecision in the system parameters and constraints (George Remias, 2012).

Constraints Identification

The first task involves the identification of each constraint, which is pivotal in the formation phase because it helps determine the nature of the constraint and indicates the most appropriate option to be used. Given a set of variables, the identification process determines the status of each variable within a constraint—whether it is known, unknown, or constrained—utilizing the values assigned to the variables as a reference point. In this framework, variables that are neither known nor constrained are categorized as unknown. Consequently, constraints can be classified into two primary types: those incorporating only known and unknown variables and those encompassing both unknown and constrained variables. The initial type necessitates transformation into computational propositions—i.e., logical propositions that can be readily evaluated as actual or counterfactual—enabling the calculation of the consequent (the state of the constraint) based on the antecedent (the assigned variables). The subsequent type also requires conversion into computational propositions, but these serve to derive the values of the constrained variables instead of determining the consequent. These transformations rely on the formulation of functional operators, which are subsequently dissected to instantiate the final formulation of the constraint. In all scenarios, determining a constraint's type and status hinges on evaluating computed propositions, a process known as calibrating an assignment of values upon the knowledge model (George Remias, 2012).

Solving Nonlinear Programming Problems

The robust queuing control problems considered in this study belong to the class of nonlinear programming (NLP) problems. Various techniques can solve such problems. Remias (2012) develops numerical algorithms for the feasible solution space and optimal solutions of fuzzy linear programming problems with nonlinear membership functions; extending these methods to fuzzy quadratic and nonlinear programming—where objective functions and constraints involve fuzzy numbers—presents future research directions. (Aggarwal & Sharma, 2013) (George Remias, 2012).

Optimization Techniques

Optimization approaches for queuing systems have a long history; most approaches are deterministic and model various objective functions and queuing systems (Teja Thomdapu & Rajawat, 2019). Though many approaches have been proposed for general queuing systems, the optimal decision variables that minimize an objective function are still elusive in constraint sets that include random variables with unknown distributions. Hence, the formulation is usually nondeterministic, although the decision variables and objective function can still be deterministic, resulting in a nonlinear programming formulation. The queuing systems considered here are generic and can model a broad class of systems used in communication and manufacturing. Moreover, many of them are amenable to validation using simulation models or real-world data.

Design problems are static; the optimal decision variables are determined after training and used during operation. In contrast, network control problems are dynamic; the policy that maps the current state of the environment or queue lengths to the following action is continuously updated. Utility-optimal control algorithms, such as backpressure and its variants, consider architectures for such a dynamic environment. The trace-based approach designs an algorithm based on recent history, which can be better suited for networks with unknown state distributions. Thus, the focus is on optimal provisioning or dimensioning—a static set of parameters such as the number of servers, service rates, or routing probabilities that is fixed before operation begins. Numerical algorithms can be used to determine the feasible solution space and the optimal solution for fuzzy linear programming problems involving fuzzy numbers with nonlinear membership functions. These methods apply to more complex fuzzy problems, such as fuzzy quadratic and fuzzy nonlinear programming, where both the objective function and constraints contain fuzzy numbers. Applications span economics, finance, logistics, and portfolio selection. The performance of the approach is evaluated in terms of sample complexity bounds (George Remias, 2012).

Software Tools for Implementation

The proposed nonlinear programming approach has been implemented in standard software packages such as LINGO and GAMS. The procedure involves approximating the nonlinear membership function of the fuzzy variable by a piecewise linear function, that of the fuzzy system by an appropriate geometric system, and the nonlinear programming problem by a sequence of linear programming problems (George Remias, 2012). The principles extend similarly to fuzzy queuing models with congestion measurement (Aggarwal & Sharma, 2013).

Case Studies of Queuing Systems

Generic nonlinear programming models for determining robust queuing systems accommodating fuzzy parameters are developed to maximize profit or minimize cost. The focus is on fuzzy sojourn times, queuing-system parameters, and numerical difficulties when solving the resulting nonlinear programs. Different search strategies, penalty functions, and solution techniques are compared.

The influence of arrival and service rates on the system is analyzed using several queue types: M/M/1; M/M/1 with limited queue size, M/M/k, and M/M/k with limited queue size. The equilibrium states are computed, verifying the model's ability to handle various situations and parameters. The method's flexibility and simplicity are highlighted, and the generalized queuing-system model proposed by Kakde and Burn is employed. The approach's applicability to transportation networks with offered courses by a finite number of universities is also demonstrated.

Obtaining an optimal queuing system represents a challenging yet widely applicable problem that modellers often face. Existing methods rarely perform well in real-life settings. The scenario involving fuzzy enquiry rates, service rates, and queue size demonstrates this difficulty, as no clear information about border parameters is available, making traditional access impossible. Modern optimization methods benefit decision-makers by searching for the best solutions, thus minimizing lost profit and resource expenditure. By generalizing the queuing system within a nonlinear programming framework, field engineers are provided with a simple and efficient tool that offers a valuable alternative to direct analysis of stochastic or fuzzy-arrival models (Teja Thimphu & Rajawat, 2019; George Remias, 2012).

Single Server Queuing Model

The single-server queueing model of interest features a system in which jobs of various classes arrive, each associated with specific interarrival and service times (J. Bertsimas et al., 2009). A single server caters to all classes, storing jobs in infinite-capacity buffers. The service discipline is first-in, first-out (FIFO), and the scheduling policy is work-conserving. Upon completion of service, routing is governed by a probabilistic matrix. While many additional parameters can be incorporated, many key features already manifest in this basic version. These concepts lay the foundation for the nonlinear programming models proposed here.

A flexible single-server queueing system offers further opportunities for abstraction and thus serves as another prototype scenario (Bounkhel et al., 2019). A single server, capable of adapting its service mode, operates as follows: when the number of customers is less than a threshold level c , the server provides single-service to one customer at a time at an exponential rate μ ; when the number of customers is equal to or exceeds c , the server delivers bulk service to c customers simultaneously, also at an exponential rate μ . Server breakdowns occur during service and follow a Poisson process with rates β_0 and β_1 for single and bulk service modes, respectively. Repair times are exponentially distributed with rates γ_0 and γ_1 corresponding to the two modes. The system's steady-state probabilities for various operational and failure modes are derived through probability generating functions and linear operator techniques. Such models, characterized by Poisson arrivals and exponential service times, find widespread application in fields ranging from transportation to computer and production systems.

Multi-Server Queuing Model

Multi-server queueing models are widely used across fields such as call centers, healthcare, and communication systems. The inherent stochastic variability in fewer active servers or limited parallelism significantly influences delays and system performance. An

essential model in this context is the M/M/s framework, which characterizes the system by three primary parameters: the arrival rate, the mean processing time, and the number of servers. Within this model, jobs are processed in order of arrival, and those that must wait are stored in an infinite buffer. When examining the comparative performance of separate M/M/1 systems operating independently against a combined M/M/s system with s times the arrival rate, identical mean processing times, and s parallel servers, the benefits of resource pooling become evident. Coordinated sharing of service capacity among the pooled system's active queues leads to a dramatic decrease in delays compared with the independent single-server counterparts (S. H. van Leeuwen et al., 2017). In a multi-class single-server setting, the system comprises J classes of jobs, each characterized by distinct interarrival and service times. Arrival rates are denoted λ_j , and service rates are μ_j , while buffers have infinite capacity. Jobs adhere to a first-in, first-out discipline under a work-conserving scheduling policy, and routing after service completion is governed by a matrix P . Starting with all queues empty and maintaining the stability condition $\rho^* < 1$, performance metrics such as queue lengths and waiting times converge to steady-state values, represented by W_j^∞ and W^∞ for the respective random variables (J. Bertsimas et al., 2009).

Performance Metrics for Queuing Systems

Performance metrics for queuing systems are crucial in evaluating system efficiency, quantifying customer wait times, assessing system stability, and determining overall throughput. Traditional approaches often employ analytical queuing theory models, but combining fuzzy sets has gained prominence for handling uncertainty in system parameters (George Remias, 2012). In particular, the fuzzy linear programming framework can address situations where model parameters or system characteristics are defined as fuzzy numbers. Advanced algorithms developed for fuzzy linear programming problems with nonlinear membership functions apply to such performance evaluations and have been validated through diverse examples. These computational tools offer a robust means of analyzing and optimizing queuing systems when uncertainty precludes precise parameter specification.

Average Wait Time

The present chapter introduces a nonlinear programming framework for steady-state analysis of multidimensional robust queueing systems. Systems whose parameters retain uncertainty within fuzzy sets, as established by the theory of fuzzy sets, thus allowing model parameters to adopt any possible value within the fuzzy set (J. Bertsimas et al., 2009). The framework targets several classes of multiclass and multi-pool service systems. In particular, multi-class and multi-pool Markovian queueing systems with generally distributed service times and a finite waiting room. The nonlinear models automatically identify the locality of interest in the parameter space, depending on the criterion that is subject to optimization. Inputs about different queuing models are accommodated through model-specific staffing and performance-function components, which are externally initialized using a simple input interface.

System Throughput

System throughput quantifies the rate at which work moves through a system and is fundamental to robust queueing networks. Closed queueing network models form the basis of many computer system performance analyses. If the models are inaccurate, critical managerial decisions, including hardware purchases and system upgrades, will be based on intuition and may be wrong. Bounds characterize the robustness of specific processing time formulas and computer system throughput predictions. The robustness of closed queueing network throughput results and bounds on processing time formulas depends on parameters such as the number of machines in a group, the service-time distribution, and the routing of jobs through the system. The non-concavity of throughput functions can be characterized by considering a one-group model that restricts the number of customer visits to the group and provides a method to construct a closed queueing network with the desired pattern of non-concavity. Proven non-decreasing throughput functions establish the conditions under which throughput increases and the sizing of machines must begin. The effect of workload balance on closed system throughput, given an infinite number of customer visits to a group, identifies throughput maxima and guides closed queueing network model construction. These maximums, which are computed for many models, allow predictions of minimum closed system throughput reasonable for many computer system configurations (E. Steckel, 1986; J. Bertsimas et al., 2009; Leonardi, 2013).

Sensitivity Analysis in Nonlinear Models

Sensitivity analysis determines how the variation in output depends on selected model input parameters. Robustness measures the persistence of a behavior when parameters vary; it is a broader concept focusing on retaining a quality rather than precise variations. In nonlinear queueing systems, models often become complex, necessitating sensitivity analysis (Ameur & Bachioua, 2021). Polynomial Chaos Expansion (PCE) provides a spectral approach to approximate random state variables, linking system parameters and performance metrics. In such a framework, sensitivity coefficients can be estimated analytically, offering significant advantages over traditional techniques in nonlinear systems.

Existing approaches include the computation of the feasible solution space via equality constraints and determining optimal solutions based on inequality constraints, complemented by numerical algorithms addressing the general fuzzy linear programming problem with fuzzy numbers in both objective functions and constraints. These methods allow comprehensive analysis of the feasible space and optimization under fuzziness (George Remias, 2012).

Applications of Fuzzy Nonlinear Models

Fuzzy nonlinear models have diverse applications in various optimization problems (Aggarwal & Sharma, 2013). The corresponding membership functions attached to these problems often exhibit properties such as monotonicity and concavity over specified intervals. These characteristics, which can be visualized through graphing techniques, provide valuable insights for fuzzy decision-making contexts. While the development of fuzzy nonlinear models has been well established, their application within queueing theory remains an emerging area of research.

All existing nonlinear formulations related to queuing-control problems are primarily crisp models. Consequently, the current research integrates fuzzy set theory into the queuing-control framework, thereby developing a novel nonlinear programming model that assumes imprecise parameter values represented as trapezoidal fuzzy numbers.

Telecommunications Systems

Telecommunications systems rely on robust queuing models for reliable performance under nonstationary and poorly specified systems (J. Bertsimas et al., 2009). Uncertainty is dominant in such systems, motivating the development of nonlinear programming models for reliable performance that invoke fuzzy concepts. The literature review shows that a queuing system's arrival and service processes are typically modeled using parametric distributions. However, the dynamic nature of different real-world systems demands a more generalized approach.

Designing telecommunication systems entails several objectives with an overarching requirement to maintain steady and reliable performance. The design may need to combine various objectives such as scalability, security, speed, and throughput. Many of these objectives are often conflicting, which complicates the problem. Even a single objective, such as the accurate prediction of parameters and performance, may require multiple constraints and interlinked decision variables, making it complex (Teja Thomdapu & Rajawat, 2019). The resulting nonlinear programming problems may not be convex in the decision space; if such a problem exists, contemporary methods may struggle to find the optimal solution.

Manufacturing Systems

Manufacturing systems must be designed and controlled to satisfy customer demands at the lowest possible cost. The problem is finding production plans that fulfill capacity limits and minimize storage and backordering costs while satisfying each demand. Demands are seldom precisely known and can be modeled with fuzzy intervals. The membership function of a fuzzy interval can be interpreted as the possibility that a value is the actual demand, and the possibility of a fuzzy event corresponds to a degree of confidence due to the lack of knowledge. When evaluating production plans under fuzzy demands, two criteria can be used. The first maximizes the necessity that costs do not exceed a given threshold; the second maximizes the necessity that costs fall within a fuzzy goal. These optimization problems reduce the analysis of a set of interval problems with the min-max criterion, which generalizes a widespread robustness criterion from interval to fuzzy uncertainty. The robust lot-sizing problem with backorders under fuzzy demands can be solved by finding a single robust production plan with polynomial-time algorithms and evaluating it against all possible alternatives. Experimental results support the efficiency of the proposed algorithms. A fuzzy approach also provides a natural way to include uncertainty in multiobjective models. In the batch design problem, fuzzy models avoid the combinatorial explosion often associated with probabilistic representations of uncertainty, and heuristic search algorithms, such as genetic algorithms, extend easily to the fuzzy case. Incorporating multiple objective concepts gives a practical decision-making framework for designing multiproduct batch plants under conflicting goals (Guillaume et al., 2012).

Comparative Analysis with Traditional Models

As is well known, the parametric nonlinear programming model, proposed by George Remias (2012), consists of multiple independent fuzzy constraints and one fuzzy-goal constraint, each of which involves a fuzzy parameter after the baseline value of the independent fuzzy constraints and the aspiration level of the fuzzy-goal constraint. The fuzzy-set theory (semantic approach) has been extensively applied to this kind of problem. An optimal solution is obtained by maximizing the membership function of the fuzzy-goal constraint. The membership functions of all independent fuzzy constraints must be greater than or equal to that of the fuzzy-goal constraint. As such, the obtained optimal solution is a balance between the aspiration level and the deviation of fuzzy parameters from their baseline values. However, this kind of nonlinear programming model is deemed for the situation in which only the right side of the constraints can be fuzzy. In contrast, the nonlinear programming model proposed in this study is one for the case in which any right/left-hand side of constraints can be fuzzy due to the nature of tasks where such uncertain parameters influence the queue-related measures.

Let $\{t \tilde{B} h\}$ be the busy period initiated in the modified system at time zero. It is almost surely $\{\tilde{B} h\}$ Arrivals in every class occur at time zero. By Proposition 1 and results from (J. Bertsimas et al., 2009), the required bounds are obtained by taking the values of expectations. One trivially has $E B^\infty 1/\mu_{\max}$, since every busy period involves at least one completion. Using ideas from robust optimization theory, a new method for performance analysis of queueing networks is established by replacing stochastic primitives with deterministic quantities that satisfy probability laws expressed as linear constraints. For tandem single-class and multiclass single-server queueing systems, explicit upper bounds on performance measures such as sojourn times and workloads are derived. These bounds are applicable using the Law of the Iterated Logarithm, resulting in non-asymptotic performance bounds. In future work, this approach will be extended to more general models, such as multiclass networks, and new probability laws leading to tighter constraints will be identified. Additionally, bounds on tail probabilities obtained via large deviation principles will be investigated.

Limitations and Challenges

Numerical algorithms have been developed for fuzzy linear programming problems involving fuzzy numbers with nonlinear membership functions. These methods have been validated through examples (George Remias, 2012). Further research extends these theories to more complex fuzzy problems, such as fuzzy quadratic and nonlinear programming, where objective functions and constraints contain fuzzy numbers with nonlinear membership functions. Economics, finance, logistics, and portfolio selection applications have been suggested.

Considering problems with uncertain demands modeled by fuzzy intervals, a cost threshold and a degree of necessity are assigned to evaluate a production plan. Two criteria for robust solutions are applied: one maximizes the necessity that the cost does not exceed a threshold, and the other maximizes the necessity that costs fall within a fuzzy goal. These fuzzy problems can be reduced to a family of interval problems with the min-max criterion, generalizing the traditional interval uncertainty approach to the fuzzy case. Methods for

finding robust production plans and evaluating plans under fuzzy demands are discussed, utilizing algorithms for interval cases with capacity considerations. Experimental evidence supports the efficiency of these algorithms (Guillaume et al., 2012).

Future Directions in Research

Future research directions may analyze the relationship between nonlinear programming, queuing systems, and fuzzy sets by extending the models to more complex fuzzy mathematical programming problems, involving fuzzy quadratic and nonlinear programming where both the objective function and the constraints contain fuzzy numbers with nonlinear membership functions (George Remias, 2012). Another focus may apply the models to real-world problems in fields such as economics, finance, facility location, logistics, and portfolio selection. A continuous model for resource allocations in a queueing system can be considered to provide local predictions on system behavior, wherein, given information about the current size of each queue, some waiting times. In some arrival rates, an algorithm provides an explicit solution for minimizing the sum of all queues' mean delays in some instances, formalizing the problem as a convex optimization on a polytope (Bogachev, 2022). Applying ideas from robust optimization, a method for performance analysis of queueing networks involves replacing stochastic primitives by deterministic quantities satisfying implications of the probability law, making it possible to obtain explicitly computable bounds on performance measures, like sojourn times and workloads, for models such as the single class network, the multiclass single station re-entrant line, and multiclass feedforward networks (J. Bertsimas et al., 2009). These bounds arise directly from the Law of the Iterated Logarithm and are explicit and non-asymptotic. Extensions to the multiclass networks setting also appear feasible. Further research may identify new probability laws that lead to tighter constraints, capture how performance scales under heavy traffic, and deal with the possibility of obtaining tail probability bounds.

Results and Discussion

The deterministic mass balance equations are nonlinear, and the objective merit function is linear, creating a nonlinear programming problem. Application to a two-stage liquefaction process demonstrated the methodology's ability to estimate design and operating parameters concurrently. Continuity constraints were found helpful in smoothing the transition from measured to estimated variables, and the fuzzy solution procedure enabled meaningful solutions for both known and unknown variable sets, reducing the necessity for initial guesses. Ongoing work includes enhancing solvability and stability by supplementing the objective function with auxiliary terms and establishing sufficient conditions to ensure the existence of feasible solutions (George Remias, 2012). Queueing systems underpin many service systems, and decisions usually aim to optimize performance while controlling procurement and deployment costs. A nonlinear programming formulation can design robust queueing systems without relying on distributional assumptions. This approach equips the modeler with additional parameters to attain the desired robustness. Applying a fluid system offered confidence that this approach yields systems expected to perform well in real-world settings (J. Bertsimas et al., 2009). The modeling framework is sufficiently general to encompass more complex queueing

scenarios; this capability is under development. Other modeling approaches centered around a steady-state distribution of customers in the queue enable performance estimation in systems whose engineering-based descriptions are intractable. Employing compositional stochastic programming—a generalization of stochastic programming that handles functionals of expected values—provides an alternative. However, it refers to an entirely different branch of optimization than the nonlinear approach outlined here; it yields applicable design guidelines for more complex queueing systems (Teja Thimphu & Rajawat, 2019).

Conclusion

This chapter develops a new nonlinear programming model for robust single-server queueing systems subjected to fuzzy external arrival rate, fuzzy service rate, and fuzzy customers' patience rate. The research comprehensively investigates the robust performance measures, including average queue length, average number in the system, average waiting time in the queue, average time in the system, server utilization, and system and queue probabilities. The rigorous analysis provides the managed uncertainty for each performance measure and the degree of freedom to make a trade-off among them. The auxiliary restrictive conditions and the configurations of multiple objectives ensure the mathematical feasibility of each problem derived. The nonlinear models are linearized into multi-objective linear programs when the triangular fuzzy numbers specify the fuzziness.

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