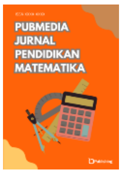

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An In-Depth Study of Cauchy-Euler Differential Equations and Their Numerical Solutions Using MATLAB

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Abstract: The Cauchy–Euler differential equation, distinguished by its dependence on the independent variable through variable coefficients, represents an essential category of linear differential equations with broad applications across mathematics, physics, and engineering. The present work provides a comprehensive exploration of both homogeneous and non-homogeneous forms of these equations, focusing on their analytical treatments as well as numerical approaches for cases in which explicit closed-form solutions are not easily attainable. Through the use of a logarithmic substitution, reckonings with mutable constants can be distorted into constant-coefficient reckonings, allowing the application of classical strategies such as the characteristic polynomial method and the approach of undetermined coefficients. Illustrative examples are included to show the derivation of both general and particular solutions, covering situations with repeated or complex roots. In addition, the study incorporates numerical techniques—particularly MATLAB's ode45 solver—to approximate solutions of Cauchy–Euler equations, especially for non-homogeneous problems and systems subject to initial conditions. These implementations highlight the adaptability and efficiency of numerical solvers in scenarios where analytical methods are difficult or infeasible. By combining analytical and computational methodologies, this work provides an integrated framework for addressing a wide range of differential equations encountered in scientific and engineering applications. It also emphasizes the enduring significance of Cauchy–Euler equations and the crucial role of computational platforms such as MATLAB in modern differential equation analysis.

Keywords: Cauchy-Euler Equation, Differential Equations, Logarithmic Transformation, Homogeneous Equations, Non-Homogeneous Equations, MATLAB, Numerical Methods, Ode45, Characteristic Equation, Analytical Solutions

Introduction

The investigation of differential equations constitutes a fundamental aspect of applied mathematics, serving as a basis for diverse applications in physics, engineering, computer science, and related disciplines. Within the broad spectrum of such equations, the Cauchy–Euler equation occupies a distinctive role owing to its structural properties and the extensive theoretical background it offers. Commonly referred to as the Euler–Cauchy or equidimensional comparison, it epitomizes a class of rectilinear everyday discrepancy reckonings in which the constants are spoken as controls of the independent variable. The over-all representation of this reckoning in the nth order can be printed as:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x),$$

where a0,a1,…,an are constant coefficients and f(x) is a given continuous function representing the non-homogeneous term. This form contrasts with constant-coefficient linear differential equations by incorporating variable coefficients, posing unique challenges in both analytical and numerical solution methods.

A fundamental technique for addressing the Cauchy-Euler equation is the use of variable substitution, such as x=et which transforms the original equation with variable coefficients into one with constant coefficients. This transformation simplifies the problem and allows the application of well-established methods for constant-coefficient linear differential equations. However, such substitutions are not always straightforward or efficient, especially in applied contexts where computational resources and time are limited. This encourages the consideration of alternative strategies, ranging from direct analytical procedures to numerical techniques. The latter have become increasingly important in recent years, driven by advancements in computational efficiency and the availability of specialized software. Among these, MATLAB has established itself as a leading platform for the numerical treatment of differential equations. Its built-in solvers, such as ode45, provide reliable and versatile tools for approximating solutions to a broad spectrum of ordinary differential equations, particularly when closed-form solutions are not attainable. The present work seeks to deliver an in-depth examination of the Cauchy–Euler differential equation, highlighting both its theoretical foundations and its practical numerical resolution. The discussion begins with classical analytical approaches, including the use of the characteristic polynomial to address homogeneous cases and the technique of hesitating constants for non-homogeneous equations. The scope is then extended to higher-order forms, illustrating the adaptability and general applicability of these techniques.

The study then turns to numerical solution approaches within the MATLAB environment. It demonstrates the implementation of the Cauchy–Euler equation and its extensions, emphasizing practical aspects such as the handling of initial conditions, numerical stability, and accuracy of results. A series of illustrative examples are provided to show the progression from exact analytical solutions to approximate numerical ones, thereby creating a clear link between theoretical analysis and computational practice.

In addition, the discussion highlights the wide-ranging relevance of these solution strategies across scientific fields. The Cauchy–Euler equation emerges in diverse contexts, including algorithmic analysis in computer science, modeling of engineering systems, and the study of partial differential equations in mathematical physics. Mastering both exact and numerical approaches strengthens the capacity to address real-world problems where analytical solutions are either difficult or unattainable.

In summary, this work not only consolidates the theoretical framework of Cauchy–Euler equations but also provides researchers and practitioners with effective computational techniques for their numerical treatment, thereby enhancing understanding and expanding applications across multiple scientific and engineering disciplines.

The Use of the Cauchy Equation and Euler Equation with the Main Differential Equation

The Cauchy-Euler equation, also referred to as the Cauchy or Euler equation, is a specific type of linear differential equation characterized by constant coefficients. This form is especially useful when solving equations in which the independent variable appears raised to a power. The method simplifies such equations by introducing a substitution—typically converting them into linear differential equations with constant coefficients, which are significantly easier to solve. The key benefits and applications of this method include:

- The Cauchy–Euler method is chiefly efficient for reckonings where the self-governing variable seems in power form. By smearing the substitution= $x\ ^{t}e$. such power terms are converted into exponential expressions, which greatly simplifies the solution process.
- By employing an appropriate substitution, the Cauchy–Euler equation can be rewritten as a linear differential equation with constant coefficients. This reformulation makes it possible to apply classical and well-developed techniques for solving constant-coefficient equations in a more straightforward manner.
- Differential equations serve as a fundamental tool for describing a wide range of physical systems in both science and engineering. The Cauchy–Euler approach provides an effective means of simplifying and solving such equations, thereby facilitating the analysis of complex real-world phenomena.
  - Analysis of Special Functions:

    Solutions of Cauchy–Euler equations frequently involve exponential or power functions, which play a vital role in both mathematical theory and physical applications. (7)
  - Eigenvalue Problems:

    Cauchy–Euler equations also arise in the context of eigenvalue problems, where solving specific differential equations is essential for determining eigenvalues and corresponding eigenvectors. (4)
  - It is important to note that, although the Cauchy–Euler method is highly effective for certain types of differential equations, its applicability is limited. The method is most suitable when the equation contains power terms of the independent variable; in cases where this condition is not satisfied, alternative analytical or numerical techniques may be required.

Introduction to the Cauchy-Euler Method

The Cauchy-Euler method provides a systematic approach for solving linear differential equations with constant coefficients, specifically of the form:

$$a_n x_n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x^1 y^{`} + a_0 y = 0 \qquad (1)$$

where $a_0, a_1, \ldots, a_n$ are real endless coefficients and $x \in R$, They are frequently among the first examples of higher-order homogeneous linear differential equations with variable coefficients presented in undergraduate courses. Typically, the discussion begins with second-order Cauchy–Euler equations. The apposite form for its answer is y = t r where r is a limit to be resolved. Substituting y with t r in the Cauchy–Euler reckonings yields the typical polynomial whose roots control the forms of the over-all solution This same method can be approved over to resolve the advanced order Cauchy–Euler reckonings.

The normal form of the Cauchy–Euler reckoning plays a vital role in the wider theory of lined difference equations, chiefly since of its direct request to the Fourier way for cracking partial difference equations—especially the second-order form.

Within the outline of delivery theory, R. P. Kanwal [3] categorizes the answers to ordinary similar linear differential reckonings of the type:

$$a_n(x)y^{(n)}\ (x) + a_{n-1}(x)y^{(n-1)}\ (x) + \cdots + a_1(x)y^{'}\ (x) + a_0\ (x)y(x) = 0,$$

In this form, the coefficient functions a0(x), a1(x), …, an(x) are infinitely differentiable, where x∈R. An answer is considered traditional if it is continuously differentiable at smallest n epochs, such that the reckoning holds in the usual (strong) sense. If the differentiability conditions are not fully satisfied, and the equation holds only in a weak or distributive sense, a weak solution exists. A distributive solution is a remarkable supply that satisfies the equation in the generalized puny sense. This method ultimately alters the Cauchy-Euler equation into a reckoning with constant constants, allowing for a closed-form key.

$$z(t)\left(1-\frac{d}{dt}\right)\frac{d}{dt}=x^2y''(x) \quad ,\frac{dz}{dt}=z(t), \quad xy'(x)=y(x)$$

Consider the second-order Cauchy–Euler equation

$$.0 = cy + 'bxy + ''ax^2y$$

To justify the substitutions used in solving this equation, it is necessary to establish a general identity, from which the derivative transformations naturally follow. This identity can be proven rigorously using **mathematical induction**. In the induction step, the **chain rule** is applied, which states that if = yy(x) and = xx(t)

$$.\frac{dt}{dx}\cdot\frac{dy}{dt}=\frac{dy}{dx}$$

This approach provides a systematic foundation for the direct replacement of derivative terms in the Cauchy–Euler equation, ensuring that the transformed equation maintains equivalence with the original

Example (1-1):

(By what means to Solve a Cauchy-Euler Equation) Demonstration the answer details for the reckoning

$$ax^2 y'' + bxy' + cy = 0$$

verifying general solution

$$,c_2 x^{1/2-} \sin\left(\frac{\sqrt{5}}{2}\ln|x|\right) + c_1 x^{1/2-} \cos\left(\frac{\sqrt{5}}{2}\ln|x|\right) = y(x)$$

Solution: The characteristic equation $2r(r-1) + 4r + 3 = 0$ can be obtained as follows:

$2x^2y'' + 4xy' + 3y = 0$

$2x^2r(r-1)x^{r-2} + 4xrx^{r-1} + 3x^r = 0$

$2r(r-1) + 4r + 3 = 0$

$2r^2 + 2r + 3 = 0 \Longrightarrow r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$

Application of the Cauchy-Euler Equation in Differential and Non-Differential Equations

In many scientific and engineering applications, problem-solving often involves

N

Nth-order linear differential equations, among which the Cauchy–Euler equation frequently appears. This equation is commonly encountered in diverse contexts, including the examination of processer algorithms—particularly in investigative the behavior of quicksort and the structure of search trees—as well as in physics and manufacturing, such as once solving Laplace's reckoning in polar organizes. Outside these areas, Cauchy–Euler equations rise in numerous additional areas of science. Numerous solution methods are available for such reckonings, and the Cauchy–Euler technique is documented in the works as a well-established and actual tactic.

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0,$$

with $a_0, a_1, \ldots, a_n$coefficients is called the same Euler-Cauchy reckoning of order n. 1 As exposed in Adjuncts A and C, by familiarizing a new mutable

$$z = \ln\ |x|,$$

The Cauchy–Euler equation can be converted into an

n

nth-order linear differential equation with constant coefficients, for which the solution methods are well-known and extensively analyzed. In this discussion, however, we focus on deriving the solutions of the Euler–Cauchy equation directly, without resorting to the variable substitution approach described earlier.

It is also possible to solve the inhomogeneous Euler-Cauchy differential equation, where the right-hand side is replaced by a known function of x, as shown below:

$$,f(x) = a_0 y + \frac{dy}{dx} a_1 x + \cdots + \frac{d^{n-1} y_{n-1}}{{}^{n-1} dx} a_{n-1} x + \frac{d^n y_n}{{}^n dx} a_n x$$

Analogous to linear difference equations by constant coefficients, the method of undetermined coefficients can be successfully employed in situations where the non-homogeneous term

f(x) assumes certain specific forms. For clarity and simplicity, the present discussion concentrates on second-order Euler–Cauchy differential equations, though the extension of these methods to higher-order cases is relatively straightforward. (see Appendix C).

The Cauchy-Euler Equation in the Context of Homogeneous, Non-Homogeneous, and Exponential Linear Differential Equations

Equations that are same in the self-governing variable continue invariant below a scaling alteration of the form x→αx x, where α\alphaα is any non-zero continuous. In general, such equations can be expressed as:

$$,0 = F\left(y,\, x\frac{dy}{dx},\, x^2 \frac{d^2 y}{{}^2 dx}\right)$$

The substitution $z(y) = xy'_x$

leads to a first order equation

$$,0 = (z - F(y, z, z'z$$

Some differential equations that are not directly separable in their original form can be transformed into separable ones by applying an appropriate change of the dependent variable. This method is specifically used for a class of equations referred to as Euler homogeneous differential equations.

Definition:

An Euler homogeneous differential equation can be expressed as:

$$y'(t) = F(\frac{y(t)}{t}).$$

Here, F is a function of ttt and yyy, but depends only on the ratio $\frac{y}{t}$ Such a function is scale invariant, meaning its value remains unchanged under the transformation:

$$y \rightarrow cy, t \rightarrow ct,$$

$$F(\frac{(cy)}{(ct)}) = F(\frac{y}{t}). \quad [3]$$

This property makes these equations also known as scale-invariant differential equations.

(b) Scale-invariant purposes are a exact case of same functions of grade n , which satisfy the stuff:

$$f(ct, cy) = c^n f(y, t).$$

An example of a homogeneous function can be found in thermodynamics. Consider the energy E of a gas, which be contingent on the entropy S and the capacity V of the scheme. In this setting, energy is a same function of grade one, such that:

$$,\mathbb{R} \ni c\, E(S, V) \quad \text{for all } c = E(cS, cV)$$

Example (1-2): Show that the meanings f1 and f2 are similar and find their grade

$$^3ty + {}^2t^2y = t^3y^3, \quad f_2(t, y) + {}^5ty + {}^2t^4y = f_1(t, y)$$

Solution: The purpose f1 is similar of gradation 6, since

$$(^3t^3y + {}^5ty + {}^2c^6(t^4y = {}^3c^6t^3y + {}^5c^6ty + {}^2c^6t^4y = f_1(ct, cy)$$

$$,c^6 f_1(t, y) = c^6 \left(t^4y^2 + ty^5 + t^3y^3\right) = f_1(ct, cy)$$

And the sum of the supremacies of t and y on each stretch is 4.

Example (1-3) : Protest that $(t - y)y' - 2y + 3t + \frac{y^2}{t} = 0$ is an Euler homogeneous equation.

Solution: Rewrite the equation in the standard form

Consider the first-order differential equation

$$\frac{^2y}{t} - 3t - 2y = \frac{dy}{dt}(y - t)$$

By solving for $\frac{dy}{dt}$, the equation can be expressed explicitly as

$$\frac{\frac{^2y}{t} - 2y - 3t}{t - y} = \frac{dy}{dt}$$

So the function f in this case is given by

$$\frac{\frac{^2y}{t} - 2y - 3t}{t - y} = f(t, y)$$

This function is scale invariant, since numerator and denominator are same of the same grade, $n = 1$ in this case

$$\frac{\frac{2y}{t} - 2y - 3t}{t - y} = f(t, y)$$

yields

$$f(t, y) = \frac{c(2y - 3t - \frac{2y}{t})}{c(t - y)} = \frac{\frac{2(cy)}{ct} - 2cy - 3ct}{ct - cy} = f(ct, cy)$$

So, the difference equation is Euler same. We now write the reckoning in the form $y' = F(y/t)$ Since the numerator and denominator are same of degree $n = 1$, we increase them by "1" in the procedure $(1/t)^{1/}(1/t)^{1}$ that is

$$\frac{\frac{2y}{t} - 2y - 3t}{t - y} = \frac{dy}{dt}$$

Allocate the factors (1/t) in numerator and denominator, and we get

$$\left(\frac{y}{t}\right) F = \frac{dy}{dt} \quad \Rightarrow \quad \frac{^{2}\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right) 2}{\frac{y}{t} - 1} = \frac{dy}{dt}$$

Where

$$\frac{^{2}\left(\frac{y}{t}\right) - 3 - \left(\frac{y}{t}\right) 2}{\frac{y}{t} - 1} = \left(\frac{y}{t}\right) F$$

So, the reckoning is Euler same and it is written in the normal form

Linear Dependence and Linear Independence

If $y_1, y_2, y_3 \ldots y_n$ a sate of connected function by $[a, b]$

And if $c_1y_1 + \cdots + c_ny_n = 0$ such that $c_1 = 0, c_2 = 0, \ldots, c_n = 0$ are constant then the site of the function is $y_1, y_2, \ldots, y_n$ is called Linearly independent functions.

If $c_1y_1 + c_2y_2 +, \ldots, +c_ny_n$ such that $c_1, c_2, \ldots, c_n$ are not all of it equal to zero ,Then $y_1, y_2, \ldots, y_n$ It is called a connected set

If $y_1, y_2, \ldots, y_n$ site of connected functions on the $[a, b]$ and differentiable on $(n - 1)$ we can defined the locator Wronskian

$$w[y_1, y_2, \ldots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If there are $y_1, y_2, \ldots, y_n$ sets of functions connected on the closed in terval [a,b] and differentiable (n − 1) once, then is verified the equation :

$$p_0(x)y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

If $w[y_1, y_2, \dots, y_n]$ then the site of the function is $y_1, y_2, \dots, y_n$ is called Linearly independent.
Example .
If the site of the function $y_1 = \sin x$, $y_2 = \cos x$ Then the value of Wronskian of this function is
since

$$w[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

Linearly independent functions.
Example (2-1)
If the site of the function
$y_1 = x^2 + 2x$, $y_2 = x^3 + x$, $y_3 = 2x^3 - x^2$

Then the value of Wronskian of this function is

$$w[y_1, y_2, y_2] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1^* & y_2^3 & y_3^3 \end{vmatrix} = \begin{vmatrix} x^2 + 2x & x^3 + x & 2x^3 - x^2 \\ 2x + 2 & 3x^2 + 1 & 6x^2 - 2x \\ 2 & 6x & 12x - 2 \end{vmatrix} = 0$$

If $w[y_1, y_2, y_{n3}]$ formerly the site of the purpose is y_1, y_2, y_{n3} is named a linked set

A Solution to Homogeneous Linear Equations of Order n With Constant Coefficients.

Let $y = e^{\lambda x}$ is Answer to the reckoning:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

So the Auxiliary equation

$$0 = {}_n a + a_{n-1}\lambda + \cdots + {}^{n-1} a_1 \lambda + {}^n a_0 \lambda$$

We get the roots:

$$\lambda_1, \lambda_2 \dots, \lambda_3$$

We get different solutions depending on the relationships between those root.

(1) If
$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$
Then General solution:
$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$

(2) If all the roots are real, one of the roots is repeated k times such that

$$. {}_k\lambda = /_{k+1}\lambda = /_n\lambda = \cdots = {}_3\lambda = {}_2\lambda = {}_1\lambda$$

The general solution

$$^{\lambda_n x} c_n e + \cdots + {}^{\lambda_{k+1} x} c_{k+1} e + {}^{\lambda x} c_k x^{k-1} \Big) e + \cdots + {}^{2} c_3 x + c_2 x + {}_1 c \Big) = y$$

(3) if the root $\lambda_1 = \lambda_2 = \lambda_3 = \alpha + i\beta$

Thin the general solution of this root is

$$\Big[c_6 x^2) \sin(\beta x) + c_5 x + {}_4 c) + c_3 x^2) \cos(\beta x) + c_2 x + {}_1 e^{\alpha x} \Big[(c = y$$

Example (2-2): fined the general solution

$$y'' - 4y' + 4y = 0$$

Solution

$$(D^2 - 4D + 4)y = 0$$

Let $y = e^{\lambda x}$ is the solution thin the Auxiliary equation is :

$$\lambda^2 - 4\lambda + 4 = 0$$

And the root is

$\lambda_1 = 2 \quad , \quad \lambda_2 = 2$

Thin the general solution

$$y = (c_1 + c_2 x)e^{2x}$$

Example (2-3): fined the general solution

$$y''' - 4y' = 0$$

Solution:

$$(D^3 + 4D)y = 0$$

Let $y = e^{\lambda x}$ is the solution thin the Auxiliary equation is:

$\lambda^3 - 4\lambda = 0$

and the roots is $\lambda(\lambda^2 - 4) = 0 \quad \therefore \lambda_1 = 0, \lambda_2 = 2i, \lambda_3 = -2i$

and the general solution is:

$$y = c_1 e^{0x} + (c_2 \cos 2x + c_3 \sin 2x) e^{0x}$$

$$\therefore y = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

A Resolution of Non - Homogeneous Linear Calculations of Order n With Constant Coefficients Is :

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x) \cdots (*) \qquad , a_0 \neq 0$$

Such that a_0, a_{n-1}, a_n are constant , f(x) A incessant purpose on ranges.

differential operator D Symbolic image of the reckoning (*) is :

$$\Phi(D)y = f(x) \cdots (1)$$

Such that $\Phi(D)$ polynomial function from degree n in D.

$$\Phi(D) = a_0 D^n + \cdots a_{n-1} D + a_n$$

And to solve the general solution follow the condition:

(1) find the solve of the homogeneous equations such that

$$\Phi(D)y = 0$$

We denote the subsequent resolution by y_H such that y_HIt contents the homogeneous reckoning only.

(2) We find a special solution, symbolized by the y_p, that satisfies equation

(3) There is a general solution y_G such that $y_G = y_H + y_P$ then is satisfies, equation , we must prove that y_G it is fulfilled the equation then :-

$$\because \Phi(D)y = 0$$

$$\therefore \Phi(D)y_H = 0$$

y_p is fulfilled the equation:

$$\Phi(D)y = f(x)$$

And to proof y_G is fulfilled the equation

$$\Phi(D)y_G = \Phi(D)\left[y_H + y_p\right] = \Phi(D)y_H + \Phi(D)y_p = 0 + f(x) = f(x)$$

Methods for finding the special solution of the non-homogeneous linear differential equation

Solving Simultaneous Linear Differential Equations with Constant Coefficients

Suppose we have two differential equations with constant coefficients and aim to determine a function that simultaneously satisfies both. This can be accomplished by applying algebraic techniques such as elimination and successive substitution. In this way, one can first solve a difference equation with continuous constants to obtain its general solution, which can then be used to derive the general solutions for the remaining variables.

Note

In this part we assume that $x = x(t), y = y(t)$ z and hence

$$D_x = x' = \frac{dx}{dt}$$

$$D^2 x = x'' = \frac{d^2 x}{dt^2}$$

$$D_y = y' = \frac{dy}{dt}$$

$$D^2 y = y'' = \frac{d^2 y}{dt^2}$$

Example (2-9)**:**

Solve the following system of differential equations

$$4x' - y' + 3x = \text{sint}$$

$$x' + y = \text{cost}$$

Solution

Using the effects method, such that we $D=\frac{d}{dt}$ can rewrite the system as

$$(4D+3)x-D_y=\sin t\ \ldots(1)$$
$$D_x+y=\cos t\ \ldots(2)$$

By solving equations (1) (2), we get:

$$(D^2-D-2)x=4t$$
$$\lambda^2+4\lambda+3=0$$
$$\Rightarrow(\lambda+3)(\lambda+1)=0$$
$$\therefore\quad \lambda=-3\quad,\quad \lambda=-1$$

To solve it, we use the Eyler formula,$x=e^{\lambda t}$ so we impose the auxiliary

$$x=c_1e^{-3t}+c_2e^{-t}$$
$$\because x'+y=\cos t$$
$$\therefore y=\cos t-x'$$

Therefore, the solution is the homogeneous equation by taking the form

$$y=\cos t+3c_1e^{-3t}+c_2e^{-t}$$

Example (2-10):

Solve the following system of differential equations

$$x'=x+y$$
$$y'=2x+4t$$

When $x(0)=2,\ \ y(0)\ =1$

Using the effects method, such that we $D=\frac{d}{dt}$ can rewrite the system as

$$(D-1)x-y=0\ \cdots(1)$$

$$-2x+D_y=4t\cdots(2)$$

By solving equations (1) (2), we

$$(D^2-D-2)x=4t$$
$$x'=x+y$$
$$y=x'-x$$

In compensation we get :

$$y=-2c_1e^{-t}+c_2e^{2t}+2t-3$$

When x(0)=2 then

$$c_1+c_2=1$$

And when y(0)=1 then

$$-2c_1+c_2=4$$

$\therefore c_1=-1,c_2=2$

Hence your solution is :

$$x=-e^{-t}+2e^{2t}-2t+1$$
$$y=2e^{-t}+2e^{2t}+2t-3$$

Differential Equations with Non-Constant Coefficients Are Transformed into Differential Equations with Constant Coefficients.

We focus on differential equations with variable coefficients and demonstrate how, through an appropriate alteration, these equations can be rehabilitated into differential reckonings with constant numbers.

First: The Cauchy-Eyler equation

And take the picture :

$$A_0x^ny^{(n)} + A_1x^{n-1} - y^{(n-1)} + \cdots + A_{n-1}\,xy' + A_ny = f(x)$$

Where $A_0, A_1, \ldots, A_{n-1}, A_n$ are fixed numbers, we can solve them using substitution$X = e^t$ andthus :

$$\frac{dx}{dt} = e^t = x$$

$$\frac{dy}{dt} = \frac{dy}{dx}\text{so that} \quad x \quad , \frac{dy}{dt}\frac{1}{x} = \frac{dt}{dx}\cdot\frac{dy}{dt} = \frac{dy}{dx}$$

Permission :

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right).\frac{dt}{dx}$$

$$= \frac{1}{x}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Also :

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) = \frac{d}{dt}\left(e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right).\frac{dt}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{1}{x}\left(-2e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + e^{-2t}\left(\frac{d^3y}{dt^3} - \frac{d^2y}{dt}\right)\right)$$

$$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dt^2} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)$$

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$$

Example (2-11):

Discovery the answer of the difference reckoning

$$x^2y'' + 2xy' - 6y = 0$$

The solution :

By putting $x = e^t$ then :

$$x\frac{dy}{dx}=\frac{dy}{dt}$$

And the :

$$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$$

So the difference equation goes into :

$$\frac{d^2y}{dt^2}+\frac{dy}{dt}-6y=0$$

Using the Eller method, we assume that $y=e^{\lambda t}$ and we obtain the auxiliary equation :

$$\lambda^3+\lambda-6=0$$

$$(\lambda+3)(\lambda+2)=0$$

$$\lambda=-3\ ,\lambda=2$$

Therefore, the general solution takes pictures

$$y=c_1e^{-3t}+c_2e^{2t}=c_1x^{-3}+c_2x^2$$

Example(2-11):

Finding the solution to the differential equation

$$x^3y'''+6x^2y''+6xy'=\sin(\ln x)$$

Solution :

According to the:$x=e^t$

$$x\frac{dy}{dx}=\frac{dy}{dt}$$

And:

$$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$$

And :

$$x^3\frac{d^3y}{dx^3}=\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+2\frac{dy}{dt}$$

So the differential equation turns into :

$$\sin t=\frac{dy}{dt}2+\frac{d^2y}{{}^2dt}3+\frac{d^3y}{{}^3dt}$$

Using effects, we container rewrite the reckoning in the procedure :

$$(D^3+3D^2+2D)y=\text{sint}$$

To solve it, we assume that $y=y_H+y_p$ and thus :

$$(D^3+3D^2+2D)\left(y_H+y_p\right)=\text{sint}$$

We have two differential equations :

$$(D^3+3D^2+2D)y_H=0\quad \dots(1)$$

$$(D^3+3D^2+2D)y_p=\sin t\dots\quad(2)$$

To solve equation (1), we use the Eller method, so we assume that $y=e^{\lambda t}$ and we obtain the auxiliary equation

$\lambda^3+3\lambda^2+2\lambda=0$

$$\Rightarrow \lambda(\lambda+1)(\lambda+2)=0$$

$$\lambda_1 = 0 \quad , \lambda_2 = -1 \, , \qquad \lambda_3 = -2$$

Hence the answer of the finite is as

$$^{2t-}c_3e + {}^{t-}c_2e + {}_1c = Hy$$

To solve equation (2), we use Laws of Influences

$$y_p = \frac{1}{D^3+3D^2+2D}(\sin t) = \frac{1}{D(D^2+3D+2)}(\sin t)$$

$$y_p = \frac{1}{D(-1+3D+2)}(\sin t)$$

$$y_p = \frac{1}{D(3D+1)}\sin t = \frac{1}{3D^2+D}\sin t$$

$$y_p = \frac{1}{(-3+D)}(\sin t) = \frac{1}{-3+D}.\frac{-3+D}{-3+D}\sin t$$

Therefore, the general solution takes the form

$$y = y_H + y_P$$

$$y = c_1 + c_2e^{-t} + c_3e^{-2t} + \frac{-1}{10}(3\sin t + \cos t)$$

$$y = c_1 + \frac{C_2}{x} + \frac{C_3}{x^2} - \frac{1}{10}(3\sin\ln x + \cos\ln x)$$

Cauchy-Euler reckoning in the over-all case:

It takes the form:

$$A_n y + {}'b)y + A_{n-1}(ax + \cdots + {}^{n-1}b)^{n-1}y + A_1(ax + {}^{n}b)^n y + A_0(ax = f(x)$$

Where

A_0 , A_1, ... , A_{n-1} and $A_0 \neq 0$

We can solve it using substitution $ax + b = e'$ Consequently

$$\frac{dx}{dt} = \frac{e'}{a} = \frac{ax+b}{a}$$

$$\Rightarrow \frac{dt}{dx} = \frac{a}{ax+b}$$

Permission:

$$\frac{dy}{dt} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{a}{ax+b}.\frac{dy}{dt}$$

$$\frac{(ax+b)\ (dy)}{dx} = a\,\frac{dy}{dt}$$

Also:

$$\frac{d^2(y)}{d^2x} = \frac{d}{dx}\left(\frac{a}{ax+b}.\frac{dy}{dt}\right)$$

$$= \frac{a^2}{ax+b}\left(-\frac{e^{-t}dy}{dt} + \frac{e^{-t}d^2y}{dt^2}\right)$$

$$= \frac{(ax+b)^2 d^2\,y}{dx^2} = a^2\left(\frac{d^{2y}}{dt^2} - \frac{dy}{dt}\right)$$

and also :

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\left(\frac{a^2}{(ax+b)^2}\right)\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) = a^2\frac{d}{dt}\left(e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right).\frac{dt}{dx}$$

$$= \frac{a^3}{ax+b}\left(-2e^{-2t}\left(\frac{d^{2y}}{dt^2} - \frac{dy}{dt}\right) + e^{-2t}\left(\frac{d^3y}{dt^3} - \frac{d^{2y}}{dt^2}\right)\right)$$

$$= \frac{a^3}{(ax+b)^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} - 2\frac{dy}{dt}\right)$$

$= (ax+b)^3\frac{d^3y}{dx^3} = a^3(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} - 2\frac{dy}{dt})$

Example (2-12):

Find the solution to the differential equation:

$(x+1)y'' - 3(1+x)y' + 4y = (x+1)^2$

Solution:

Put

$$1 + x = e'$$

Ephemeral

$$(1+x)\frac{dy}{dx} = \frac{dy}{dt}$$

And :

$$\frac{dy}{dt} - \frac{d^2y}{{}^2dt} = \frac{d^2y_2}{{}^2dx}(x+1)$$

Thus, the discrepancy reckoning turns hooked on:

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^{2t}$$

Using the effects, we can write the equation:

$$(D^2 - 4D + 4)y = e'$$

And to solve here we suppose

$$y = y_H + y_P$$

Consequently:

$$(D^2 - 4D + 4)(y_H + y_P) = e^{2t}$$

So we have two differential equations:

$$(D^2 - 4D + 4)y_H = 0 \ldots (1)$$

$$(D^2 - 4D + 4)y_P = e^{2t} \ldots (2)$$

To crack the reckoning, we use the Eller process (1) So suppose that

$$y = e^{xt}$$

We get the auxiliary equation:

$$\lambda^2 - 4\lambda + 4 = 0$$
$$(\lambda - 2)^2 = 0$$
$$\lambda_1 = \lambda_2 = 2$$

So the solution to the homogeneous equation takes the form:

$$y_H = e^{2t}(c_1 + c_2\, t)$$

To solve equation (2), we use the laws of :

$$y_P = \frac{1}{(D-2)^2} e^{2t} = \frac{t^2}{2!} e^{2t} = \frac{t^2 e^{2t}}{2}$$

So the general solution takes the form of:

$$y = y_H + y_P$$
$$y = e^{2t}(c_1 + c_2\, t) + \frac{t^2 e^{2t}}{2}$$
$$y = (x+1)^2\big(C_1 + C_2(\ln(x+1))\big) + \frac{\ln 2(x+1)}{2}$$

Homogeneous and Non-Homogeneous Cauchy-Euler Differential Equations by Using the Methods in MATLAB

1. **For homogeneous equations:**

```
% The main equation
function yprime = myODE(t, y)
   % Homogeneous Cauchy-Euler equation
   yprime = -2*t*y;
end

% Initial conditions
t0 = 0;
y0 = 1;

% Time points to compute the solution
tspan = [0 10];

% Solve the equation using ode45
[t, y] = ode45(@myODE, tspan, y0);
```

```
% Plot the solution curve
figure;
plot(t, y, 'b','LineWidth',1.5);
xlabel('Time (t)');
ylabel('Response y(t)');
title('Time Evolution of the Solution');
grid on;
```

2. For non-homogeneous equations:

```
  % The main equation
function yprime = myODE(t, y)
   % Non-homogeneous Cauchy-Euler equation
   yprime = -2*t*y + sin(t);
end

% Initial conditions
t0 = 0;
y0 = 1;

% Time points to compute the solution
tspan = [0 10];

% Solve the equation using ode45
[t, y] = ode45(@myODE, tspan, y0);

% Display the results
plot(t, y);
xlabel('t');
ylabel('y');
```

In both cases, the main equation is defined in my ODE function, and the initial conditions t0 and y0 are specified. Then, the time period over which the solution is to be computed is defined. ode45 is called using the equation function, the initial conditions, and the time points to compute the solution. Finally, the results are plotted using the plot function.

You can customize the code based on your specific Cauchy-Euler equation by modifying the main equation y prime.

Conclusion

In this study, we delve into the Cauchy-Euler differential equation, exploring both its analytical solutions and MATLAB-based numerical methods. The Cauchy-Euler equations, with their characteristic variable coefficients, present unique challenges distinct from those of linear difference equations with continuous constants. These reckonings can be distorted into constant-coefficient form through variable substitution techniques, such as the logarithmic transformation x = et, enabling the efficient application of classical solution methods. However, real-world problems often require numerical solutions, especially for complex inhomogeneous terms or high-order equations, where closed-form solutions can be cumbersome or difficult to obtain. MATLAB's numerical solvers, particularly ode45, have proven to be powerful and flexible tools for approximating the solutions of these differential equations with high accuracy and computational efficiency. The examples presented in this article demonstrate the seamless transition between analytical and numerical methods and highlight how numerical solutions complement theoretical analysis. Furthermore, MATLAB's flexibility in handling initial conditions and various functional forms makes it particularly useful for researchers and engineers working with differential equations in real-world applications. Ultimately, mastering analytical and numerical methods for the Cauchy-Euler equations not only deepens theoretical understanding but also broadens its practical applications in physics, engineering, computer science, and other scientific fields. This integrated approach enables practitioners to effectively model and solve complex problems, thereby advancing academic research and technological development.

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